Wadi Alshatti University Journal of Pure and Applied Sciences

مجلة جامعة وادي الشاطئ للعلوم البحتة والتطبيقية

Volume 3, No. 2, July-December 2025 Online ISSN: 3006-0877 المجلد 3، الاصدار 2، يوليو - ديسمبر 2025

RESEARCH ARTICLE APPLIED MECHANICAL ENGINEERING

Internally Balanced Spatial Elasticity tensor

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ARTICLE HISTORY

Received 25 April 2025
Revised 26 May 2025
Accepted 31 May 2025
Online 01 June 2025

KEYWORDS:

Principal stretches;
Hyperelasticity;
Internal balance;
Spatial elasticity tensor

ABSTRACT

An ongoing research has applied arguments of calculus of variations to the portions of the deformation gradient multiplicative decomposition. This internally balanced treatment has a condensed material elasticity tensor that is formulated in terms of decomposed principal stretches. This work provides an explicit formulation to determine the condensed spatial elasticity tensor. The explicit formulation introduces an alternative procedure that is equivalent to pushing forward of the condensed material elasticity tensor. A demonstration is presented using an internal balance logarithmic constitutive model. A case study of simple shearing verifies the correctness of the new formulation.

ممتدة المرونة المكانية المتوازنة داخليا

أشرف حدوش[1]

الملخص

يطبّق هذا البحث براهين حساب التفاضل والتكامل للتغيرات على أجزاء التحلل المضاعف لتدرج التشوه. تحتوي هذه المعالجة المتوازنة داخليًا على موتر مرونة المادة المكثف الذي تمت صياغته بدلالة التمددات الرئيسية المتحللة. يوفر هذا العمل صياغة صريحة لتحديد موتر المرونة المكانية المكثفة. تقدم الصيغة الصريحة إجراءً بديلًا يكافئ الدفع إلى الأمام لمشد مرونة المادة المكثفة. يتم تقديم شرح توضيحي باستخدام نموذج تكويني لوغاريتمي للتوازن الداخلي. كما حققت دراسة حالة من صحة الصيغة الجديدة على قص بسيط.

الكلمات المفتاحية:

التمددات الرئيسية
فرط المرونة
التوازن الداخلي
ممتدة المرونة المكانية

Introduction

Finite deformation often applies deformation gradient multiplicative decomposition

$$F = \hat{F}\breve{F} \qquad (1)$$

That is implemented to model advance aspects of deformation in abroad spectrum of applications [1-6]. A new treatment of hyperelasticity [7,8] applies the arguments of calculus of variations to both decomposed portions. This provides the stress equations of equilibrium and an internally balanced equation which is used to determine the decomposed portions of $\boldsymbol{F}$. Recently, the material elasticity tensor is formulated using principal stretches of decomposed portions in the reference configuration [9]. Then, the spatial elasticity tensor is obtained by push forward procedure [10]. But, it is preferred to develop an explicit form of spatial elasticity tensor to facilitate its implementation in deformed configuration [11]. Therefore, the main interest of this paper it to develop the formulations of the internal balance spatial elasticity tensor. An internal balance constitutive model is introduced. The calculation of spatial elasticity tensor for simple shearing load is chosen to verify the correctness of new formulations.

Internally Balanced Formulations

The hyperelastic internal balance formulations using principal stretches [12] is briefly reviewed in this section. Principal stretches λ_a of $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$ are multiplicatively decomposed such as:

$$\lambda_a = \hat{\lambda}_a\breve{\lambda}_a \qquad (2)$$

Where $\hat{\lambda}_a$ and $\breve{\lambda}_a$ are the principal stretches of $\widehat{\boldsymbol{C}} = \widehat{\boldsymbol{F}}^T\widehat{\boldsymbol{F}}$ and $\breve{\boldsymbol{C}} = \breve{\boldsymbol{F}}^T\breve{\boldsymbol{F}}$, respectively. In this manuscript all subscript indices are equal to 1,2,3. The constitutive model has the form of $\Phi(\hat{\lambda}_a, \breve{\lambda}_a)$. The principal component of second Piola – Kirchhoff stress S_a is given by:

$$S_a = \frac{\hat{\lambda}_a}{\lambda_a^2}\frac{\partial\Phi}{\partial\hat{\lambda}_a} = \frac{1}{\lambda_a^2}\frac{\partial\Phi}{\partial\ln\hat{\lambda}_a} \qquad (3)$$

The principal component of the internal balance tensor Ψ_a is given as:

$$\Psi_a = \frac{1}{\breve{\lambda}_a}\frac{\partial\Phi}{\partial\breve{\lambda}_a} - \frac{\hat{\lambda}_a}{\lambda_a^2}\frac{\partial\Phi}{\partial\hat{\lambda}_a} = \frac{1}{\lambda_a^2}\left(\frac{\partial\Phi}{\partial\ln\breve{\lambda}_a} - \frac{\partial\Phi}{\partial\ln\hat{\lambda}_a}\right) \qquad (4)$$

It is worth to mention here that the second Piola – Kirchhoff stress $\boldsymbol{S}$ is derived by minimizing the total energy w.r.t. $\boldsymbol{C}$ at

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https://doi.org/10.63318/waujpasv3i2_11

fixed $\breve{\boldsymbol{C}}$ while the internal balance tensor $\boldsymbol{\Psi}$ is achieved by minimizing the total energy w.r.t. $\breve{\boldsymbol{C}}$ at fixed $\boldsymbol{C}$.

The internal balance treatment has a condensed material elasticity tensor [13]. It is reformulated using principal stretches in [9]. Using Voigt notation, the condensed material elasticity tensor $\boldsymbol{\mathcal{C}}$ in principal directions is structured as:

$$[\boldsymbol{\mathcal{C}}] = \begin{bmatrix} [\boldsymbol{\mathcal{C}_D}] & 0 \\ 0 & [\boldsymbol{\mathcal{C}_O}] \end{bmatrix} \tag{5}$$

Where:

$$[\boldsymbol{\mathcal{C}_D}] = \left[\frac{\partial S_a}{\partial E_b}\right] - \left[\frac{\partial S_a}{\partial \breve{E}_b}\right]\left[\frac{\partial \Psi_a}{\partial \breve{E}_b}\right]^{-1}\left[\frac{\partial \Psi_a}{\partial E_b}\right] \tag{6}$$

$$[\boldsymbol{\mathcal{C}_O}] = \left[\frac{S_a - S_b}{2(E_a - E_b)}\right] - \left[\frac{\partial S_a}{\partial \breve{E}_b}\right]\left[\frac{\partial \Psi_a}{\partial \breve{E}_b}\right]^{-1}\left[\frac{\Psi_a - \Psi_b}{2(E_a - E_b)}\right] \tag{7}$$

Where $\boldsymbol{I}$ is the identity tensor. The terms E_a and $\breve{E}_a$ are the principal components of $\boldsymbol{E}$ and $\breve{\boldsymbol{E}}$, respectively, that are $2\boldsymbol{E} = \boldsymbol{C} - \boldsymbol{I}$, $2\breve{\boldsymbol{E}} = \breve{\boldsymbol{C}} - \boldsymbol{I}$ and. It is recommended to review [9] for further details about the definition of the terms in (6) and (7).

The spatial elasticity tensor has several definitions. It is given as a push forward of $\boldsymbol{\mathcal{C}}$ [10, 14], then the condensed spatial elasticity tensor $\boldsymbol{\mathcal{D}}$ becomes:

$$\mathcal{D}_{abcd} = \lambda_a\lambda_b\lambda_c\lambda_d\mathcal{C}_{abcd} \tag{8}$$

It can also be defined as a Piola transformation of $\boldsymbol{\mathcal{C}}$ [15, 16] that defines the condensed spatial elasticity $\boldsymbol{\mathcal{H}}$ as:

$$\mathcal{H}_{abcd} = J^{-1}\lambda_a\lambda_b\lambda_c\lambda_d\mathcal{C}_{abcd} \tag{9}$$

Where $J = \det \boldsymbol{F} = \lambda_1\lambda_2\lambda_3$. Observe that $\boldsymbol{\mathcal{D}}$ is often used with Kirchhoff stress $\boldsymbol{\tau}$ while $\boldsymbol{\mathcal{H}}$ is used with Cauchy stress $\boldsymbol{\sigma}$ hence $\boldsymbol{\sigma} = J^{-1}\,\boldsymbol{\tau}$ and $\boldsymbol{\mathcal{H}} = J^{-1}\boldsymbol{\mathcal{D}}$.

Condensed Spatial elasticity

An internal balance explicit formulation of condensed spatial elasticity tensor $\boldsymbol{\mathcal{D}}$ is developed following the leads of Crisfield [10]. The principal component of Kirchhoff stress τ_a is defined by pushing forward of (3)**:**

$$\tau_a = \lambda_a^2\, S_a = \hat{\lambda}_a \frac{\partial \Phi}{\partial \hat{\lambda}_a} = \frac{\partial \Phi}{\partial \ln \hat{\lambda}_a} \tag{10}$$

The derivative of S_a w.r.t. E_b at fixed $\breve{E}_b$ can be achieved it terms of τ_a by differentiating $S_a = \tau_a/\lambda_a^2$ such as:

$$\begin{aligned}\frac{\partial S_a}{\partial E_b} &= \frac{1}{\lambda_a^2}\frac{\partial \tau_a}{\partial E_b} - \frac{2}{\lambda_a^4}\tau_a\delta_{ab} \\ &= \frac{1}{\lambda_b\lambda_a^2}\frac{\partial \tau_a}{\partial \lambda_b} + \frac{\hat{\lambda}_b}{\lambda_b^2\lambda_a^2}\frac{\partial \tau_a}{\partial \hat{\lambda}_b} - \frac{2}{\lambda_a^4}\tau_a\delta_{ab} \\ &= \frac{1}{\lambda_b^2\lambda_a^2}\left(\frac{\partial \tau_a}{\partial \ln \lambda_b} - \frac{\partial \tau_a}{\partial \ln \hat{\lambda}_b}\right) - \frac{2}{\lambda_a^4}\tau_a\delta_{ab}\end{aligned} \tag{11}$$

Whre δ_{ab} is Kronecker delta. The derivative of S_a w.r.t. $\breve{E}_b$ at fixed E_b becomes:

$$\begin{aligned}\frac{\partial S_a}{\partial \breve{E}_b} = \frac{1}{\lambda_a^2}\frac{\partial \tau_a}{\partial \breve{E}_b} &= \frac{1}{\breve{\lambda}_b\lambda_a^2}\frac{\partial \tau_a}{\partial \breve{\lambda}_b} + \frac{\hat{\lambda}_b}{\breve{\lambda}_b^2\lambda_a^2}\frac{\partial \tau_a}{\partial \hat{\lambda}_b} \\ &= \frac{1}{\breve{\lambda}_b^2\lambda_a^2}\left(\frac{\partial \tau_a}{\partial \ln \breve{\lambda}_b} - \frac{\partial \tau_a}{\partial \ln \hat{\lambda}_b}\right)\end{aligned} \tag{12}$$

Noting that (11) and (12) are achieved by making use of $2E_b = \lambda_a^2 - 1$ and $2\breve{E}_b = \breve{\lambda}_a^2 - 1$. Further details concerning the derivatives of the decomposed principal stretches $\hat{\lambda}_a$ and $\breve{\lambda}_a$ w.r.t. E_a and $\breve{E}_b$ can be reviewed in [9].

The condensed spatial elasticity $\boldsymbol{\mathcal{D}}$ is structured in Voigt notation as:

$$[\boldsymbol{\mathcal{D}}] = \begin{bmatrix} [\boldsymbol{\mathcal{D}_D}] & 0 \\ 0 & [\boldsymbol{\mathcal{D}_O}] \end{bmatrix} \tag{13}$$

Pushing forward of (6) provides the matrix $[\boldsymbol{\mathcal{D}_D}]$

$$\begin{aligned}[\boldsymbol{\mathcal{D}_D}] &= [\lambda_a^2\lambda_b^2\mathcal{C}_D] \\ &= \left[\lambda_a^2\lambda_b^2\,\frac{\partial S_a}{\partial E_b}\right] \\ &\quad - \left[\lambda_a^2\lambda_b^2\,\frac{\partial S_a}{\partial \breve{E}_b}\right]\left[\frac{\partial \Psi_a}{\partial \breve{E}_b}\right]^{-1}\left[\frac{\partial \Psi_a}{\partial E_b}\right]\end{aligned} \tag{14}$$

That is reformulated by using (2), (11) and (12) such as:

$$\begin{aligned}[\boldsymbol{\mathcal{D}_D}] &= \left[\lambda_b\frac{\partial \tau_a}{\partial \lambda_b} + \hat{\lambda}_b\frac{\partial \tau_a}{\partial \hat{\lambda}_b} - 2\tau_a\delta_{ab}\right] \\ &\quad - \left[\hat{\lambda}_b^2\left(\breve{\lambda}_b\frac{\partial \tau_a}{\partial \breve{\lambda}_b} - \hat{\lambda}_b\frac{\partial \tau_a}{\partial \hat{\lambda}_b}\right)\right]\left[\frac{\partial \Psi_a}{\partial \breve{E}_b}\right]^{-1}\left[\frac{\partial \Psi_a}{\partial E_b}\right] \\ &= \left[\frac{\partial \tau_a}{\partial \ln \lambda_b} + \frac{\partial \tau_a}{\partial \ln \hat{\lambda}_b} - 2\tau_a\delta_{ab}\right] \\ &\quad - \left[\hat{\lambda}_b^2\left(\frac{\partial \tau_a}{\partial \ln \breve{\lambda}_b} - \hat{\lambda}_b\frac{\partial \tau_a}{\partial \ln \hat{\lambda}_b}\right)\right]\left[\frac{\partial \Psi_a}{\partial \breve{E}_b}\right]^{-1}\left[\frac{\partial \Psi_a}{\partial E_b}\right]\end{aligned} \tag{15}$$

Note that the last two terms of (15) are not affected by pushing forward procedure and they continue to have their original definition:

$$\frac{\partial \Psi_a}{\partial E_b} = \frac{1}{\lambda_b}\frac{\partial \Psi_a}{\partial \lambda_b} + \frac{\hat{\lambda}_b}{\lambda_b^2}\frac{\partial \Psi_a}{\partial \hat{\lambda}_b} = \frac{1}{\lambda_b^2}\left(\frac{\partial \Psi_a}{\partial \ln \lambda_b} + \frac{\partial \Psi_a}{\partial \ln \hat{\lambda}_b}\right) \tag{16}$$

$$\frac{\partial \Psi_a}{\partial \breve{E}_b} = \frac{1}{\breve{\lambda}_b}\frac{\partial \Psi_a}{\partial \breve{\lambda}_b} - \frac{\hat{\lambda}_b}{\breve{\lambda}_b^2}\frac{\partial \Psi_a}{\partial \hat{\lambda}_b} = \frac{1}{\breve{\lambda}_b^2}\left(\frac{\partial \Psi_a}{\partial \ln \breve{\lambda}_b} + \frac{\partial \Psi_a}{\partial \ln \hat{\lambda}_b}\right) \tag{17}$$

The matrix $[\mathcal{D}_O]$ is achieved by pushinf forward of (7)

$$\begin{aligned}[\boldsymbol{\mathcal{D}_O}] = [\lambda_a^2\lambda_b^2\mathcal{C}_O] &= \left[\frac{\lambda_a^2\lambda_b^2(S_a - S_b)}{2(E_a - E_b)}\right] \\ &\quad - \left[\lambda_a^2\lambda_b^2\frac{\partial S_a}{\partial \breve{E}_b}\right]\left[\frac{\partial \Psi_a}{\partial \breve{E}_b}\right]^{-1}\left[\frac{\Psi_a - \Psi_b}{2(E_a - E_b)}\right]\end{aligned} \tag{18}$$

That is manipulated by (10) to become:

$$[\mathcal{D}_O] = \left[\frac{\lambda_b^2\tau_a - \lambda_a^2\tau_b}{\lambda_a^2 - \lambda_b^2}\right] - \left[\lambda_a^2\lambda_b^2\frac{\partial S_a}{\partial \breve{E}_b}\right]\left[\frac{\partial \Psi_a}{\partial \breve{E}_b}\right]^{-1}\left[\frac{\Psi_a - \Psi_b}{2(E_a - E_b)}\right] \tag{19}$$

Calculating $[\boldsymbol{\mathcal{D}_O}]$ requires careful treatment. Note that the principal components Ψ_a is set to zero $\Psi_1 = \Psi_2 = \Psi_3 = 0$ to determine the decomposition of (2) that leads to:

$$\left[\frac{\Psi_a - \Psi_b}{2(E_a - E_b)}\right] = 0 \tag{20}$$

This simplifies (19) to:

$$[\mathcal{D}_O] = \left[\frac{\lambda_b^2\tau_a - \lambda_a^2\tau_b)}{\lambda_a^2 - \lambda_b^2}\right] \tag{21}$$

However, the case of $E_a = E_b$ gives $\tau_a = \tau_b$ coupled with $\Psi_a = \Psi_b = 0$ leading to dividing zero by zero. To avoid that, the first and the fourth terms of (19) are calculated using the l'Hôpital rule. Applying l'Hôpital rule on the latter term gives:

$$\begin{aligned}\lim_{E_a \to E_b}\frac{\Psi_a - \Psi_b}{2(E_a - E_b)} &= \frac{1}{2}\left(\frac{\partial \Psi_a}{\partial E_a} - \frac{\partial \Psi_b}{\partial E_a}\right) \\ &= \frac{1}{2\lambda_a}\left(\frac{\partial \Psi_a}{\partial \lambda_a} - \frac{\partial \Psi_b}{\partial \lambda_a}\right) + \frac{\hat{\lambda}_a}{2\lambda_a^2}\left(\frac{\partial \Psi_a}{\partial \hat{\lambda}_a} - \frac{\partial \Psi_b}{\partial \hat{\lambda}_a}\right)\end{aligned} \tag{22}$$

Concerning the first term of (21), the l'Hôpital rule is applied prior to push forward procedure. Let us start by applying l'Hôpital rule gives:

$$\lim_{E_a \to E_b}\frac{S_a - S_b}{2(E_a - E_b)} = \frac{1}{2}\left(\frac{\partial(\tau_a/\lambda_a^2)}{\partial E_a} - \frac{\partial(\tau_b/\lambda_b^2)}{\partial E_a}\right) \tag{23}$$

$$= \frac{1}{2}\left(\frac{1}{\lambda_a^3}\frac{\partial\tau_a}{\partial\lambda_a} + \frac{\hat\lambda_a}{\lambda_a^4}\frac{\partial\tau_a}{\partial\hat\lambda_a} - \frac{2\tau_a}{\lambda_a^4}\right) - \frac{1}{2}\left(\frac{1}{\lambda_a\lambda_b^2}\frac{\partial\tau_b}{\partial\lambda_a} + \frac{\hat\lambda_a}{\lambda_a^2\lambda_b^2}\frac{\partial\tau_b}{\partial\hat\lambda_a}\right)$$

Then pushing forward is performed:

$$\lim_{E_a\to E_b}\frac{\lambda_b^2\tau_a - \lambda_a^2\tau_b}{\lambda_a^2-\lambda_b^2} = \lambda_a^2\lambda_b^2\lim_{E_a\to E_b}\frac{S_a - S_b}{2(E_a-E_b)} = \frac{1}{2}\left(\lambda_a\frac{\partial\tau_a}{\partial\lambda_a} + \hat\lambda_a\frac{\partial\tau_a}{\partial\hat\lambda_a} - 2\tau_a\right) - \frac{1}{2}\left(\lambda_a\frac{\partial\tau_b}{\partial\lambda_a} + \hat\lambda_a\frac{\partial\tau_b}{\partial\hat\lambda_a}\right) \quad (24)$$

Logarithmic Material model

A simple logarithmic constitutive model is given in [10], see equation 13.151, as:

$$W = \frac{\mu}{2}(\lambda_1^2+\lambda_2^2+\lambda_3^2) - \mu(\ln\lambda_1 + \ln\lambda_2 + \ln\lambda_3) \quad (25)$$

Where μ is a material positive parameter. In analogy to (25), an internal balance constitutive model is proposed:

$$\Phi = \frac{\mu(1+\beta)}{2}\left(\hat\lambda_1^2+\hat\lambda_2^2+\hat\lambda_3^2\right) - \mu(1+\beta)\left(\ln\hat\lambda_1+\ln\hat\lambda_2+\ln\hat\lambda_3\right) + \frac{\mu(1+\beta)}{2\beta}(\check\lambda_1^2+\check\lambda_2^2+\check\lambda_3^2) - \frac{\mu(1+\beta)}{\beta}\left(\ln\check\lambda_1+\ln\check\lambda_2+\ln\check\lambda_3\right) \quad (26)$$

Where β is a material positive parameters that is tunes the weight of the multiplicative decomposed portions.

A systematic procedure is followed to compute the internally balance spatial condensed elasticity tensor $\boldsymbol{\mathcal{D}}$**.** It is started by determining the essential derivatives and principal components such as:

$$\frac{\partial\Phi}{\partial\lambda_a} = 0,\ \frac{\partial\Phi}{\partial\hat\lambda_a} = \frac{\bar\mu\left(\hat\lambda_a^2-1\right)}{\hat\lambda_a},\ \frac{\partial\Phi}{\partial\check\lambda_a} = \frac{\bar\mu\left(\check\lambda_a^2-1\right)}{\beta\check\lambda_a}$$

$$S_a = \frac{\bar\mu\left(\hat\lambda_a^2-1\right)}{\lambda_a^2},\ \tau_a = \bar\mu\left(\hat\lambda_a^2-1\right),$$

$$\Psi_a = \frac{\bar\mu}{\check\lambda_a^2}\left(\frac{\left(\check\lambda_a^2-1\right)}{\beta} - \left(\hat\lambda_a^2-1\right)\right)$$

$$\frac{\partial S_a}{\partial\lambda_b} = \frac{-2\bar\mu\left(\hat\lambda_a^2-1\right)\delta_{ab}}{\lambda_a^3},\ \frac{\partial S_a}{\partial\hat\lambda_b} = \frac{2\bar\mu\hat\lambda_a\delta_{ab}}{\lambda_a^2},\ \frac{\partial S_a}{\partial\check\lambda_b} = 0 \quad (27)$$

$$\frac{\partial\tau_a}{\partial\lambda_b} = 0,\quad \frac{\partial\tau_a}{\partial\hat\lambda_b} = 2\bar\mu\hat\lambda_a\delta_{ab},\quad \frac{\partial\tau_a}{\partial\check\lambda_b} = 0$$

$$\frac{\partial\Psi_a}{\partial\lambda_b} = 0,\quad \frac{\partial\Psi_a}{\partial\hat\lambda_b} = \frac{-2\bar\mu\hat\lambda_a\delta_{ab}}{\check\lambda_a^2},\quad \frac{\partial\Psi_a}{\partial\check\lambda_b} = \frac{2\bar\mu\delta_{ab}}{\check\lambda_a^3}$$

Where $\bar\mu = \mu(1+\beta)$. Next, the matrix $[\mathcal{D}_D]$ is determined by calculating the matrices presented in (15) and making use of (2) to obtain

$$\left[\lambda_b\frac{\partial\tau_a}{\partial\lambda_b} + \hat\lambda_b\frac{\partial\tau_a}{\partial\hat\lambda_b} - 2\tau_a\delta_{ab}\right] = 2\bar\mu[\delta_{ab}] = 2\bar\mu\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\left[\hat\lambda_b^2\left(\check\lambda_b\frac{\partial\tau_a}{\partial\check\lambda_b} - \hat\lambda_b\frac{\partial\tau_a}{\partial\hat\lambda_b}\right)\right] = 2\bar\mu\left[\hat\lambda_a^4\delta_{ab}\right] = 2\bar\mu\begin{bmatrix}\hat\lambda_1^4&0&0\\0&\hat\lambda_2^4&0\\0&0&\hat\lambda_3^4\end{bmatrix}$$

$$\left[\frac{\partial\Psi_a}{\partial\breve E_b}\right] = 2\bar\mu\left[\frac{\left(1+\hat\lambda_a^2\right)\hat\lambda_a^4\delta_{ab}}{\lambda_a^4}\right] = 2\bar\mu\begin{bmatrix}\frac{\left(1+\hat\lambda_1^2\right)\hat\lambda_1^4}{\lambda_1^4}&0&0\\0&\frac{\left(1+\hat\lambda_2^2\right)\hat\lambda_2^4}{\lambda_2^4}&0\\0&0&\frac{\left(1+\hat\lambda_3^2\right)\hat\lambda_3^4}{\lambda_3^4}\end{bmatrix}$$

$$\left[\frac{\partial\Psi_a}{\partial E_b}\right] = -2\bar\mu\left[\frac{\hat\lambda_a^4\delta_{ab}}{\lambda_a^4}\right] = -2\bar\mu\begin{bmatrix}\frac{\hat\lambda_1^4}{\lambda_1^4}&0&0\\0&\frac{\hat\lambda_2^4}{\lambda_2^4}&0\\0&0&\frac{\hat\lambda_3^4}{\lambda_3^4}\end{bmatrix} \quad (28)$$

$$[\boldsymbol{\mathcal{D}_D}] = 2\bar\mu\begin{bmatrix}\frac{1+\hat\lambda_1^2+\hat\lambda_1^4}{1+\hat\lambda_1^2}&0&0\\0&\frac{1+\hat\lambda_2^2+\hat\lambda_2^4}{1+\hat\lambda_2^2}&0\\0&0&\frac{1+\hat\lambda_2^2+\hat\lambda_2^4}{1+\hat\lambda_3^2}\end{bmatrix}$$

The matrix $[\boldsymbol{\mathcal{D}_O}]$ as defined in (21) becomes:

$$[\boldsymbol{\mathcal{D}_O}] = \left[\frac{\lambda_b^2\tau_a - \lambda_a^2\tau_b)}{\lambda_a^2-\lambda_b^2}\right] = \bar\mu\begin{bmatrix}\frac{c_0}{\lambda_1^2-\lambda_2^2}&0&0\\0&\frac{c_1}{\lambda_2^2-\lambda_3^2}&0\\0&0&\frac{c_2}{\lambda_1^2-\lambda_3^2}\end{bmatrix} \quad (29)$$

Where $c_0 = \lambda_2^2\left(\hat\lambda_1^2-1\right) - \lambda_1^2\left(\hat\lambda_2^2-1\right)$, $c_1 = \lambda_3^2\left(\hat\lambda_2^2-1\right) - \lambda_2^2\left(\hat\lambda_3^2-1\right)$ and $c_2 = \lambda_3^2\left(\hat\lambda_1^2-1\right) - \lambda_1^2\left(\hat\lambda_3^2-1\right)$

Demonstration: Simple Shear

The deformation of simple shearing is governed by $\boldsymbol{F} = \boldsymbol{I} + \gamma\,\boldsymbol{e_1}\otimes\boldsymbol{e_2}$ [17] where γ is the amount of shear that is related to shear angle θ by $\gamma = \tan\theta$. At $\theta = \pi/4$, the principal stretches λ_a of $\boldsymbol{C}$ becomes:

$$\lambda_1 = 1.6180,\quad \lambda_2 = 0.6180,\quad \lambda_3 = 1 \quad (30)$$

Then, the scalar internal balance equations $\Psi_a = 0$ must be solved for given β and λ_a to determine the decomposed principal stretches (2). This coupling generates the following system of equations:

$$\lambda_1 = \hat\lambda_1\check\lambda_1,\ \lambda_2 = \hat\lambda_2\check\lambda_2,\ \lambda_3 = \hat\lambda_3\check\lambda_3,$$

$$\frac{\left(\check\lambda_1^2-1\right)}{\beta} - \left(\hat\lambda_1^2-1\right) = 0,\quad \frac{\left(\check\lambda_2^2-1\right)}{\beta} - \left(\hat\lambda_2^2-1\right) = 0,\quad \frac{\left(\check\lambda_3^2-1\right)}{\beta} - \left(\hat\lambda_3^2-1\right) = 0 \quad (31)$$

In case $\beta = 1$, an analytical solution is obtained such as:

$$\check\lambda_a = \hat\lambda_a = \sqrt{\lambda_a} \Longrightarrow \check\lambda_1 = \hat\lambda_1 = 1.2720,\quad \check\lambda_2 = \hat\lambda_2 = 0.7862,\quad \check\lambda_3 = \hat\lambda_3 = 1 \quad (32)$$

Given $\mu = 1$, the condensed material elasticity tensor (5) is calculated:

$$[\boldsymbol{\mathcal{C}_D}] = \begin{bmatrix}1.1672&0&0\\0&33.889&0\\0&0&6\end{bmatrix},\quad [\boldsymbol{\mathcal{C}_O}] = \begin{bmatrix}1.1056&0&0\\0&3.2361&0\\0&0&0.2918\end{bmatrix} \quad (33)$$

The standard procedure to obtain condensed spatial elasticity is to push forward (33) by applying (8) such as:

$$[\boldsymbol{\mathcal{D}_D}]_{stand} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 4.9443 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad [\boldsymbol{\mathcal{D}_O}]_{stand} = \begin{bmatrix} 1.1056 & 0 & 0 \\ 0 & 1.2361 & 0 \\ 0 & 0 & 0.76393 \end{bmatrix} \tag{34}$$

The condensed spatial elasticity tensor can now be explicitly determined using the developed new formulation as presented in (28) and (29):

$$[\boldsymbol{\mathcal{D}_D}]_{new} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 4.9443 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad [\boldsymbol{\mathcal{D}_O}]_{new} = \begin{bmatrix} 1.1056 & 0 & 0 \\ 0 & 1.2361 & 0 \\ 0 & 0 & 0.76393 \end{bmatrix} \tag{35}$$

Reviewing (34) and (35), the correctness of the new formulation is confirmed. It is worth to mention that the spatial elasticity tensor in base frame $\boldsymbol{\mathcal{D}^B}$ can be calculated via:

$$\mathcal{D}^B_{ijkl} = Q_{ia}Q_{jb}Q_{kc}Q_{ld}\mathcal{D}_{abcd} \tag{36}$$

Where $\boldsymbol{Q}$ is an orthogonal tensor contains the eigenvectors of $\boldsymbol{B} = \boldsymbol{FF^T}$.

Conclusion

An Explicit formulation is developed to determine the condensed spatial elasticity tensor using principal stretches. This is an equivalent procedure to standard procedure that is based on determining the condensed material elasticity tensor then applying push forward procedure. A significant development is introduced for the terms related to principal component of second Piola – Kirchhoff stress S_a while the terms related to Ψ_a kept their original definitions. The new formulation of the condensed spatial elasticity tensor becomes dependent on principal component of Kirchhoff stress τ_a in addition to Ψ_a. The formulations of material and spatial elasticity tensors are extended by introducing the derivatives in terms of natural logarithmic of principal stretches. A simple internally balanced logarithmic constitutive model is introduced. It is demonstrated that the calculated condensed spatial elasticity tensor for simple shearing using the new formulation is equal to calculated spatial elasticity tensor by standard pushing forward of material elasticity tensor which confirmed the correctness of the new formulation.

Future work

Currently, further development is in progress to introduce an explicit formulation of the spatial condensed elasticity tensor for decoupled strain energy function which will pave way to implement more advanced material models.

Author Contributions: Single author contribution
Funding: "This research received no external funding."
Data Availability Statement: "Not applicable."
Conflicts of Interest: "The author declares no conflict of interest."
Acknowledgments: "The author would like to express his appreciation to Palestine Technical University – Kadoorei."

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