Wadi Alshatti University Journal of Pure and Applied Sciences

مجلة جامعة وادي الشاطئ للعلوم البحتة و التطبيقية

Volume 4, No. 2, July-December 2026 Online ISSN: 3006-0877 المجلد 4، الاصدار 2، يوليو - ديسمبر 2026

RESEARCH ARTICLE APPLIED MATHEMATICS

On Generalized BW-Four-Recurrent Finsler Spaces and Their Role in the Geometric Modelling of Intelligent Robotic Systems

Adel Mohammed Ali Al-Qashbari[1,*], Mohammed Saeed Moqbel Abdullah[2], Mukhtar Abdulsalam Irhouma[3], Suliman Mahmoud Ahmad[3]

[1]Department of Mathematic, Faculty of Education, University of Aden, Aden, Yemen
[2]Department of Mechatronics Engineering, Faculty of Engineering and Computing, University of Science and Technology, Aden, Yemen
[3]Mechanical and Renewable Energy Engineering Department, Engineering Faculty, Wadi Alshatti University, Brack, Libya

ARTICLE HISTORY
Received 28 July 2026
Revised 31 August 2026
Accepted 07 September 2026
Online 09 September 2026

KEYWORDS
Robotic motion modelling;
Finsler geometry;
Generalized recurrent space;
Projective curvature tensor;
Intelligent systems.

ABSTRACT

This paper presents and examines a new class of Finsler spaces known as Generalized BW-Four-Recurrent Finsler spaces $G\mathcal{B}W\text{-}FRF_n$, in which a particular recurrence relation involving non-null covariant vector fields a_lmns and b_lmns is satisfied by the fourth-order Berwald covariant derivative of the projective curvature tensor W^i_{jkh}. We prove that all generalized BW-recurrent spaces are special cases of GBW-FRF_n , and we determine the necessary and sufficient conditions for the non-vanishing of several important curvature tensors, including the scalar curvature W, the curvature vector W_k, and the projective deviation tensor W^i_h. In the setting of $G\mathcal{B}W\text{-}FRF_n$ geometry, the work also provides recurrence conditions for Cartan's third and fourth curvature tensors and the h(v)-torsion tensor. A detailed analysis is conducted of the interactions between several higher-order curvature tensors under repeated Berwald differentiation. Significantly, the paper shows how $G\mathcal{B}W\text{-}FRF_n$ structures can be used as sophisticated mathematical models for intelligent robotic systems that operate in non-Euclidean, anisotropic situations. In robots operating in dynamically complicated or curved settings, such geometric frameworks can improve motion planning, adaptive control, and sensor-based navigation. The theoretical findings provide a basis for interdisciplinary applications of higher-order Finsler geometry in autonomous systems and advanced robotics.

حول فضاءات فِنْسلر المعمَّمة رباعية التكرار، ودورها في النمذجة الهندسية للأنظمة الروبوتية الذكية

عادل محمد علي القشبري[1,*]، محمد سعيد مقبل عبدالله[2]، مختار عبدالسلام ارحومة[3]، سليمان محمود أحمد[3]

الكلمات المفتاحية
نمذجة الحركة الروبوتية
هندسة فِنْسلر
الفضاءات التكرارية المعمَّمة
موتر الانحناء الإسقاطي
الأنظمة الذكية

الملخص

تقدم هذه الدراسة فئة جديدة من فضاءات فِنْسلر تُعرف باسم فضاءات فِنْسلر المعمَّمة BW-رباعية التكرار، وتبحث في خصائصها الهندسية، حيث يتحقق فيها شرط تكرار خاص يتضمن حقولًا متجهية تغايرية غير منعدمة، ويكون مرتبطًا بالمشتقة التغايرية الرابعة وفق بيروالد لموتر الانحناء الإسقاطي. وقد تم البرهان على أن جميع فضاءات BW-التكرارية المعمَّمة تمثل حالات خاصة من هذه الفئة، كما تم تحديد الشروط الضرورية والكافية لعدم انعدام عدد من موترات الانحناء المهمة، بما في ذلك الانحناء القياسي، ومتجه الانحناء، وموتر الانحراف الإسقاطي. وفي إطار هندسة هذه الفئة من فضاءات فِنْسلر، تم كذلك اشتقاق شروط تكرارية لموتري الانحناء الثالث والرابع لكارتان، بالإضافة إلى موتر الالتواء h(v). كما أُجري تحليل مفصل للتفاعلات المتبادلة بين عدد من موترات الانحناء ذات الرتب العليا تحت تأثير التكرار المتعاقب للمشتقة وفق بيروالد. وتكمن أهمية الدراسة بصورة خاصة في توضيح إمكانية توظيف هذه البنى الهندسية بوصفها نماذج رياضية متقدمة للأنظمة الروبوتية الذكية التي تعمل في بيئات غير إقليدية ومتباينة الاتجاهات. ففي الروبوتات العاملة ضمن بيئات ديناميكية معقدة أو منحنية، يمكن لهذه الأطر الهندسية أن تسهم في تحسين تخطيط الحركة، والتحكم التكيفي، والملاحة المعتمدة على المستشعرات. كما توفر النتائج النظرية أساسًا رياضيًا لتطوير تطبيقات بينية لهندسة فِنْسلر ذات الرتب العليا في الأنظمة الذاتية والروبوتات المتقدمة.

Introduction

Finsler geometry, as a natural extension of Riemannian geometry, offers a versatile framework for modelling geometric systems where direction-dependent properties are crucial. Finslerian methodologies are especially beneficial for applications in control theory, optimization, and, more recently, robotics, owing to their directional sensitivity. Traditional Euclidean or Riemannian models frequently fail to capture the complex spatial correlations and irregularities found in real-world settings in robotic systems, particularly those navigating in complicated, anisotropic, or non-homogeneous environments. Finsler spaces, on the other

*Corresponding author
https://doi.org/10.63318/waujpasv4i2_41

hand, provide more accurate modelling of such processes, especially when the underlying geometric structure displays non-symmetric features and higher-order relationships.

The examination of recurrent and generalized recurrent structures, which illuminate the stability and symmetry properties of the geometric milieu, has garnered attention due to recent advancements in Finsler geometry. Higher-order recurrence relations using curvature tensors are particularly relevant due to their capacity to model dynamic behaviour in complex systems. Higher-order recurrent Finsler spaces can be investigated in the context of robotics to assist model robot kinematics and dynamics in scenarios where motion planning and control techniques are impacted by curvature-related limits.

In this work, we offer the idea of Generalized BW-Four-Recurrent Finsler spaces, or G$\mathcal{B}$W-FRF_n. These spaces are defined by a recurrence relation involving two non-null covariant vector fields, a_{lmns} and b_{lmns}, that is satisfied by the fourth-order Berwald covariant derivative of the projective curvature tensor W^i_{jkh}. We expand the current theory to a wider context by demonstrating that the classical BW-recurrent spaces are special examples of the G$\mathcal{B}$W-FRF_n class. Additionally, we examine the behavior of a number of related curvature objects, such as the scalar curvature W, the projective deviation tensor W^i_h, and the curvature vector W_k.

We also determine the circumstances in which these tensors are non-vanishing.

Our findings also give sufficient and necessary conditions for the h(v)-torsion tensor and Cartan's third and fourth curvature tensors to recur in the G$\mathcal{B}$W-FRF_n framework.

Furthermore, we investigate the connections between compound curvature tensors when fourth-order Berwald differentiation is applied.

These insights have potential for boosting robotic modelling frameworks in addition to being theoretically exciting in differential geometry. Specifically, a strong mathematical basis for predicting the implications of anisotropic and directionally dependent limits in robot mobility is provided by the proper description of fourth-order curvature behaviours. These findings could aid develop realistic robotic simulators that can function in geometrically challenging settings, more robust navigation systems, and adaptive control techniques.

The findings of this work supply accurate analytical tools for the study of fourth-order geometric recurrence and open new avenues for a deeper knowledge of complicated geometric structures in Finsler spaces. This helps create mathematical models for intelligent robotic system simulation that are more realistic. Robots functioning in non-homogeneous situations, such as uneven terrain or places with complicated directional fields, can be defined statistically utilizing the attributes presented inside the G$\mathcal{B}$W-FRF_n structure. Additionally, this structure could boost the efficacy of adaptive motion planning and autonomous navigation algorithms in mobile or cooperative robots.

Further applied research at the nexus of differential geometry and intelligent systems may be made possible by incorporating higher-order geometric recurrence concepts into simulation and control systems, which could represent a major advancement in the design of robots capable of interacting with real-world environments.

Motivation and Objectives

In instances when typical Riemannian structures are insufficient for modelling intricate, anisotropic environments, recent advancements in differential geometry, notably in Finsler geometry, have generated important tools. The rising requirement for advanced mathematical frameworks capable of properly defining and modelling the dynamic behaviour of intelligent robotic systems acting in non-Euclidean or direction-dependent environments is the driving force behind this study. A promising basis for such applications is provided by the invention and investigation of Generalized BW-Four-Recurrent Finsler spaces G$\mathcal{B}$W-FRF_n. In addition to advancing theoretical knowledge of higher-order geometric structures, the fourth-order Berwald recurrence relations examined in this work have applications in robotics, particularly in situations requiring adaptive motion planning and navigation in irregular or non-homogeneous environments. In instances when typical Riemannian structures are insufficient for modelling intricate, anisotropic environments, recent advancements in differential geometry, notably in Finsler geometry, have generated important tools. The rising requirement for advanced mathematical frameworks capable of properly defining and modelling the dynamic behaviour of intelligent robotic systems acting in non-Euclidean or direction-dependent environments is the driving force behind this study.

From a robotics viewpoint, the properties of G$\mathcal{B}$W-FRF_n spaces may be exploited to develop more realistic models for robot kinematics and control, especially for mobile robots operating in fields with directional curvature or on shifting terrains. Furthermore, the recurrence of higher-order curvature tensors might increase the robustness of algorithms for autonomous decision-making, obstacle avoidance, and trajectory prediction. Consequently, this work has two basic goals:

1. To explore the intrinsic curvature properties of G$\mathcal{B}$W-FRF_n spaces and provide a rigorous theoretical framework for them.
2. To investigate the intrinsic to establish their promise as a mathematical foundation for forthcoming developments in intelligent systems design and robotics.

Finsler geometry has witnessed significant development in the study of non-Riemannian geometric structures, particularly in the analysis of curvature tensors, recurrence properties, and higher-order geometric spaces. The foundational framework of Finsler differential geometry established the mathematical basis for studying geometric structures, geodesics, and curvature properties in Finsler spaces [20]. Subsequent studies introduced and developed recurrence and decomposition concepts for tensor fields in generalized Finsler spaces [3,21]. Particular attention was subsequently devoted to higher-order recurrent Finsler spaces, with emphasis on curvature tensors and Cartan's curvature tensor and their associated differential properties [1,4,7]. This line of research was further extended to birecurrent Finsler spaces through the investigation of recurrence relationships involving various tensor fields [2]. Moreover, higher-order derivatives associated with the Berwald and Cartan connections were employed to investigate the structural properties of curvature tensors in Finsler spaces [5]. More advanced formulations considered the decomposition of generalized recurrent tensor fields of higher order, providing broader mathematical frameworks for characterizing complex Finsler structures [6]. In addition, the M-projective curvature tensor was investigated within generalized recurrent and birecurrent Finsler spaces, contributing to the understanding of projective curvature structures under recurrence conditions [8]. Parallel developments have addressed broader geometric and analytical aspects of Finsler geometry, including curvature and hyperbolic flows associated with natural mechanical systems [14], uniformization problems in complex Finsler geometry [15], nonlinear spectra of Finsler manifolds [16], and comparison principles in Finsler geometry [17]. The scope of Finsler geometry has also expanded toward modern physical and engineering applications, including Finsler spacetime geometry [18], generalized nonlinear and Finsler geometry in robotics [19], higher-spin theories [23], Finslerian wormhole models in modified gravity [25], and Lorentz-violation phenomena [26-30]. In parallel, tensor-based mathematical modelling has been applied to engineering problems, including spatial elasticity tensors [11], study on Generalized BK-5th Recurrent Finsler Space [12], and nonlinear stress estimation using computational methods [10]. More recently, higher-order recurrent Finsler geometry has been connected with anisotropic flow modelling and computational fluid dynamics (CFD), demonstrating the potential of higher-order geometric structures for describing complex engineering systems [9]. Despite these substantial developments, a comprehensive mathematical framework integrating higher-order generalized recurrence conditions with multiple curvature tensors and their differential relationships remains an important area of investigation. Such a framework may provide new theoretical insights into the structural properties of recurrent Finsler manifolds while also establishing a stronger mathematical foundation for advanced applications in anisotropic modelling, computational mechanics, and intelligent engineering systems.

Collectively, these investigations demonstrate a clear progression from classical recurrence conditions toward generalized, birecurrent, and higher-order tensorial structures. Nevertheless, a systematic framework capable of integrating these higher-order recurrence conditions, particularly generalized four-recurrent structures and their associated curvature tensors, remains insufficiently developed. This gap motivates the present study, which aims to establish a more comprehensive geometric framework for higher-order generalized four-recurrent Finsler spaces and to investigate the corresponding curvature and tensorial properties.

By capturing the fourth-order Berwald recurrence of the projective curvature tensor and examining its effects on different curvature components, this new structure offers insights that could advance the theoretical framework of Finsler geometry as well as its applications in physics and robotics.

Let g_{ij} and G^i_{jk} represent the components of the Finsler metric tensor and the coefficients of Berwald's connection, respectively. Regarding the directional arguments y^i, these numbers are positively homogenous of degree zero. Because of this homogeneity, the supporting elements y_i and y^i have the following fundamental identities:

a) $y_j = g_{ij}\, y^i$

b) $y_i\, y^i = F^2$ (1)

c) $\delta^k_j\, y^j = y^k$.

The standard orthogonality criterion is satisfied by the metric tensor g_{ij} and its inverse g^{ij}

$$g_{ij}\, g^{jk} = \delta^k_i = \begin{cases} 1 \ , & if \quad i = k \\ 0 \ , & if \quad i \neq k \end{cases}. \tag{2}$$

where δ^k_i is the Kronecker delta

The identities are satisfied by the (h)hv-torsion tensor C_{ijk}, which has the following definition:

$$C_{ijk} = \tfrac{1}{2}\dot{\partial}_i\, g_{jk} = \tfrac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k\, F^2 \tag{3}$$

a) $C^i_{jk}y^j = C^i_{kj}\, y^j = 0$ and b) $C_{ijk}\, y^j = 0$ (4)

A generic tensor field T^i_j Berwald covariant derivative $\mathcal{B}_k$ with regard to x^k is as follows:

$$\mathcal{B}_k T^i_j = \partial_k T^i_j - \left(\dot{\partial}_r T^i_j\right) G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk} \tag{5}$$

Additionally, the Berwald derivative disappears in the same way for the vector y^i and the fundamental metric function F.

a) $\mathcal{B}_k F = 0$ and b) $\mathcal{B}_k y^i = 0$ (6)

Nevertheless, the metric tensor g_{ij} Berwald derivative does not disappear and is provided by:

$$\mathcal{B}_k g_{ij} = -2\, C_{ijk|h}\, y^h = -2\, y^h \mathcal{B}_h C_{ijk} \tag{7}$$

The following definitions apply to the projective curvature tensor W^i_{jkh}, the projective torsion tensor W^i_{jk}, and the projective deviation tensor W^i_j (also known as Weyl-type projective tensors)

$$W^i_{jkh} = H^i_{jkh} + \frac{2\,\delta^i_j}{n+1} H_{[hk]} + \frac{2\,y^i}{n+1}\dot{\partial}_j H_{[kh]} + \frac{\delta^i_k}{n^2-1}\left(n\, H_{jh} + H_{hj} + y^r\dot{\partial}_j H_{hr} - \frac{\delta^i_h}{n^2-1}\left(n\, H_{jk} + H_{kj} + y^r\dot{\partial}_j H_{kr}\right)\right) \tag{8}$$

$$W^i_{jk} = H^i_{jk} + \frac{y^i}{n+1} H_{[jk]} + 2\left\{\frac{\delta^i_{[j}}{n^2-1}\left(n\, H_{k]} - y^r H_{k]\,r}\right)\right\} \tag{9}$$

$$W^i_j = H^i_j - H\,\delta^i_j - \frac{1}{n+1}\left(\dot{\partial}_r H^r_j - \dot{\partial}_j H\right) y^i \tag{10}$$

These tensors satisfy the following relations:

a) $W^i_{jkh}\, y^j = W^i_{kh}$, b) $W^i_{jk}\, y^j = W^i_k$, c) $W^i_{ki} = W_k$ and d) $W^i_i = W$. (11)

With regard to its final two indices, k and h, the tensor W^i_{jkh} is skew-symmetric.

Additionally, similar higher-order tensorial forms are used to define Cartan's 2nd curvature tensor (the hv-curvature tensor), P^i_{jkh}, and its associated tensors, such as the (v)hv-torsion tensor, Ricci tensors, curvature vector, scalar curvature, and the third and fourth curvature tensors R^i_{jkh} and K^i_{jkh}. In order to formulate higher-order recurrent Finsler spaces, these tensors must adhere to a number of symmetry constraints, Bianchi identities, and homogeneity relations

$P^i_{jkh} = \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jm} P^m_{kh} - C^i_{jh|k}$, or equivalent by

$P^i_{jkh} = \dot{\partial}_h\, \Gamma^{*i}_{jk} + C^i_{jr}\, C^r_{kh|s}\, y^s - C^i_{jh|k}$, or $P^i_{jkh} = C^i_{kh|j} - g^{ir} C_{jkh|r} + C^r_{jk} P^i_{rh} - P^r_{jh} C^i_{rk}$. (12)

a) $P^i_{jkh}\, y^j = P^i_{kh}$, b) $g_{ir}\, P^r_{jkh} = P_{ijkh}$, c) $g_{rp}\, P^r_{kh} = P_{kph}$, d) $P^i_{jki} = P_{jk}$ and e) $P^i_{ki} = P_k$.

a) $R^i_{jkh} = \Gamma^{*i}_{hjk} + \left(\Gamma^{*i}_{ljk}\right) G^l_h + C^i_{jm}\left(G^m_{kh} - G^m_{kl} G^l_h\right) + \Gamma^{*i}_{mk}\, \Gamma^{*m}_{jh} - k/h$,

b) $R^i_{jkh}\, y^j = H^i_{kh} = K^i_{jkh}\, y^j$, c) $R_{jk}\, y^j = H_k$, d) $R_{jk}\, y^k = R_j$, h) $R^i_i = R$, (13)

e) $R^i_{jki} = R_{jk}$, f) $H_i\, y^i = H^i_i = (n-1)\, H$ and g) $H^i_{jk}\, y^j = H^i_k$.

a) $K^i_{rkj} = \partial_j\, \Gamma^{*i}_{kr} + \left(\dot{\partial}_l\, \Gamma^{*i}_{rj}\right) G^l_k + \Gamma^{*i}_{mj}\, \Gamma^{*m}_{kr} - j/k$ and b) $K^i_{jkh} = -K^i_{jhk}$ (14)

a) $K^i_{jkh}\, y^j = H^i_{kh}$ and b) $H^i_{jkh} = K^i_{jkh} + y^m(\dot{\partial}_j\, K^i_{mkh})$

$$K^i_{jkh} + K^i_{hjk} + K^i_{khj} = 0 \tag{16}$$

a) $K^i_{jki} = K_{jk}$, b) $K_{jk}\, y^k = K_j$, c) $K_{jk}\, y^j = H_k$ and d) $K_{jk}\, g^{jk} = K$. (17)

BW-Four-Recurrent spaces that are generalized

We officially describe and characterize a new class of Finsler spaces, which we call Generalized BW-Four-Recurrent Finsler spaces $G\mathcal{B}W\text{-}FRF_n$, based on the previous research and the established relations. The fourth order Berwald covariant derivative (the outcome of applying Berwald's differential operator sequentially with respect to x^l, x^m, x^n and x^s) is represented as $\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l$. The following recurrence relation describes how this operator functions on the projective curvature tensor W^i_{jkh} :

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W^i_{jkh} = a_{lmns}\, W^i_{jkh} + b_{lmns}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_{lmn}\, \mathcal{B}_q(\delta^i_h C_{jks} - \delta^i_k C_{jhs}) y^q - 2\, b_{lms}\mathcal{B}_q(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})\, y^q - 2b_{lm}\, \mathcal{B}_s\mathcal{B}_q(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})\, y^q - 2\, b_{lns}\, \mathcal{B}_p(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})\, y^q - 2b_{ln}\mathcal{B}_s\mathcal{B}_q(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})\, y^q - 2b_{ls}\mathcal{B}_n\mathcal{B}_q(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})\, y^q - 2b_l\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}) y^q \tag{18}$$

where a_{lmns} and b_{lmns} are non-zero covariant vector fields.

Result 2.1. Every generalized $\mathcal{B}W$-recurrent Finsler space is a special case of a generalized BW-Four-Recurrent Finsler space.

Definition 2.1. A Finsler manifold in which the projective curvature tensor W^i_{jkh} satisfies the condition (18) is called a generalized-fourth-recurrent Finsler space. We denote such spaces by $G\mathcal{B}W\text{-}FRF_n$.

Transvecting (18) by y^j, and using the identities (1a), (4b), and (11a), one obtains:

$$\mathcal{B}_s\mathcal{B}_n\, \mathcal{B}_m\mathcal{B}_l\, W^i_{kh} = a_{lmns}\, W^i_{kh} + b_{lmns}\,(\delta^i_h\, y_k - \delta^i_k\, y_h) \tag{19}$$

Further contracting by y^k, and using (1.1b), (1.2), and (1.11b), yields:

$$\mathcal{B}_s\mathcal{B}_n\, \mathcal{B}_m\mathcal{B}_l\, W^i_h = a_{lmns}\, W^i_h + b_{lmns}\,(\delta^i_h\, F^2 - y_h y^i) \tag{20}$$

Accordingly, we deduce the following theorem.

Theorem 2.1. In any $G\mathcal{B}W\text{-}FRF_n$ space, the fourth-order Berwald covariant derivative of the projective torsion tensor W^i_{kh} and the projective deviation tensor W^i_h satisfy the recurrence relations given by (19) and (20), respectively.

Contracting indices i and h in (19) and (20) applying (1.1b), (2), (11c), and (11f), we obtain:

$$\mathcal{B}_s\mathcal{B}_n\, \mathcal{B}_m\mathcal{B}_l\, W_k = a_{lmns}\, W_k + (n-1)\, b_{lmns}\, y_k \tag{21}$$

$$\mathcal{B}_s\mathcal{B}_n\, \mathcal{B}_m\mathcal{B}_l\, W = a_{lmns}\, W + (n-1)\, b_{lmns}\, F^2 \tag{22}$$

These relations demonstrate that both the curvature vector W_k and the scalar curvature W must be non-zero, since their vanishing would imply $a_{lmns} = 0$ and $b_{lmns} = 0$, contradicting the definition of a $G\mathcal{B}W\text{-}FRF_n$ space.

Accordingly, we deduce the following theorem.

Theorem 2.2. In any $G\mathcal{B}W\text{-}FRF_n$ space, the curvature vector W_k and the scalar curvature W are non-vanishing, in accordance with relations (21) and (22).

Necessary and Sufficient Conditions for Generalized $\mathbf{W^i_{jkh}}$

In this section, we determine the sufficient and necessary criteria for some tensors to be generalized recurrent in the context of the generalized BW-four recurrent Finsler space $G\mathcal{B}W\text{-}FRF_n$.

Consider a 4-dimensional Riemannian manifold V_4, where the tensor of projective curvature R^i_{jkh} (Cartan's third curvature tensor) is defined as:

$$W^i_{jkh} = R^i_{jkh} + \frac{1}{3}(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h) \tag{23}$$

Applying the Berwald's fourth-order covariant derivative to equation (3.1) with respect to x^l, x^m, x^n and x^s sequentially, yields:

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, W^i_{jkh} = \mathcal{B}_s\mathcal{B}_n\, \mathcal{B}_m\mathcal{B}_l\, R^i_{jkh} + \frac{1}{3}\mathcal{B}_s\mathcal{B}_n\, \mathcal{B}_m\mathcal{B}_l\,(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h) \tag{24}$$

Substituting the generalized BW-four recurrent condition (18) into (24), we derive a complex expression involving the covariant derivatives of R^i_{jkh}, the metric tensor g_{jk}, and other associated tensors.

$$a_{lmns}\, W^i_{jkh} + b_{lmns}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2\, b_{lmn}\, \mathcal{B}_q(\delta^i_h C_{jks} - \delta^i_k C_{jhs})\, y^q - 2\, b_{lms}\mathcal{B}_q(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})\, y^q - 2\, b_{lm}\, \mathcal{B}_s\mathcal{B}_q(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})\, y^q - 2\, b_{lns}\, \mathcal{B}_p(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})\, y^q - 2\, b_{ln}\mathcal{B}_s\mathcal{B}_q(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})\, y^q - 2\, b_{ls}\mathcal{B}_n\mathcal{B}_q(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})\, y^q - 2\, b_l\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})\, y^q = \mathcal{B}_s\mathcal{B}_n\, \mathcal{B}_m\mathcal{B}_l\, R^i_{jkh} + \tfrac{1}{3}\mathcal{B}_s\mathcal{B}_n\, \mathcal{B}_m\mathcal{B}_l\,(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h)$$

By reorganizing terms and using (3.1), the equation reduces to: (25)

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, R^i_{jkh} = a_{lmns} R^i_{jkh} + b_{lmns}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_{lmn}\mathcal{B}_q(\delta^i_h C_{jks} - \delta^i_k C_{jhs})\, y^q - 2\, b_{lms}\mathcal{B}_q(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})\, y^q - 2b_{lm}\, \mathcal{B}_s\mathcal{B}_q(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})\, y^q - 2\, b_{lns}\, \mathcal{B}_p(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})\, y^q - 2b_{ln}\mathcal{B}_s\mathcal{B}_q(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})\, y^q - 2\, b_{ls}\mathcal{B}_n\mathcal{B}_q(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})\, y^q - 2b_l\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})\, y^q - \tfrac{1}{3}\, a_{lmns}\,(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h) - \tfrac{1}{3}\mathcal{B}_s\mathcal{B}_n\, \mathcal{B}_m\mathcal{B}_l\,(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h)$$

where the ellipsis denotes symmetric terms in the covariant derivatives involving b-coefficients and Cartan tensors C_{jks}.

This shows that

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, R^i_{jkh} = a_{lmns}\, R^i_{jkh} + b_{lmns}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_{lmn}\mathcal{B}_q(\delta^i_h C_{jks} - \delta^i_k C_{jhs})\, y^q + 2\, b_{lms}\mathcal{B}_q(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})\, y^q + 2\, b_{lm}\, \mathcal{B}_s\mathcal{B}_q(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})\, y^q + 2\, b_{lns}\, \mathcal{B}_p(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})\, y^q + 2\, b_{ln}\mathcal{B}_s\mathcal{B}_q(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})\, y^q + 2\, b_{ls}\mathcal{B}_n\mathcal{B}_q(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}) y^q + 2b_l\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})\, y^q \tag{26}$$

This equality holds if and only if

$$\tfrac{1}{3}\, a_{lmns}\,(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h) + \tfrac{1}{3}\mathcal{B}_s\mathcal{B}_n\, \mathcal{B}_m\mathcal{B}_l\,(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h) = 0 \tag{27}$$

Hence, we state the following:

Theorem 3.1. Within the space $G\mathcal{B}W\text{-}FRF_n$, the Cartan third curvature tensor R^i_{jkh} is generalized fourth recurrent if and only if condition (27) is satisfied.

By transecting (25) with y^j and employing identities (1a), (4b), (13b), and (13c), we obtain the Berwald covariant derivative relation for the h(v)-torsion tensor H^i_{kh}

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, H^i_{kh} = a_{lmns}\, H^i_{kh} + b_{lmns}\,(\delta^i_h\, y_k - \delta^i_k\, y_h) - \tfrac{1}{3}\, a_{lmns}\,(\delta^i_k\, H_h - y_k\, R^i_h) - \tfrac{1}{3}\, \mathcal{B}_s\mathcal{B}_n\, \mathcal{B}_m\mathcal{B}_l\,(\delta^i_k\, H_h - y_k\, R^i_h) \tag{28}$$

This shows that

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, H^i_{kh} = a_{lmns}\, H^i_{kh} + b_{lmns}\,(\delta^i_h\, y_k - \delta^i_k\, y_h)$$

Provided that

$$a_{lmns}\left(\delta^i_k H_h - y_k R^i_h\right) + \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(\delta^i_h\, y_k - \delta^i_k\, y_h\right) = 0 \qquad (29)$$

This leads to:

Theorem 3.2. The Berwald covariant derivative of order fourth of the h(v)-torsion tensor H^i_{kh} is generalized fourth recurrent if and only if condition (29) holds in $G\mathcal{B}W\text{-}FRF_n$.

Transvecting (28) by y^k, using (1b), (1c), (1a), (13g), we get

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, H^i_h = a_{lmns}\, H^i_h + b_{lmns}\left(\delta^i_h\, F^2 - y_h\, y^i\right) - \frac{1}{3}\, a_{lmns}\left(H_h\, y^i - F^2 R^i_h\right) - \frac{1}{3}\, \mathcal{B}_s\mathcal{B}_n\, \mathcal{B}_m\mathcal{B}_l\left(H_h\, y^i - F^2 R^i_h\right) \qquad (30)$$

This shows that

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, H^i_h = a_{lmns}\, H^i_h + b_{lmns}\left(\delta^i_h\, F^2 - y_h\, y^k\right) \quad .$$

If and only if

$$\mathcal{B}_s\mathcal{B}_n\, \mathcal{B}_m\mathcal{B}_l\left(H_h y^i - F^2 R^i_h\right) = a_{lmns}\left(H_h y^i - F^2 R^i_h\right) \qquad (31)$$

These it concluded the following

Theorem 3.3. In $G\mathcal{B}W\text{-}FRF_n$, Berwald covariant derivative of the orders for the deviation tensor H^i_h is generalized fourth recurrent if and only if the condition (27) holds good.

Finally, contracting indices i and h in equations (27), (28), and (30) and applying related identities (2), (1b), (4a), (13e), (13c), (13h) and (13f), lead to:

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, R_{jk} = a_{lmns}\, R_{jk} + b_{lmns}(n-1)\, g_{jk} - 2\, b_{lmn}\, \mathcal{B}_q(n-1)\, C_{jks}\, y^q - 2\, b_{lms}\, \mathcal{B}_q(n-1)\, C_{jkn}\, y^q - 2\, b_{lm}\, \mathcal{B}_s\mathcal{B}_q(n-1)\, C_{jkn}\, y^q - 2\, b_{lns}\, \mathcal{B}_p(n-1)\, C_{jkm}\, y^q - 2 b_{ln}\mathcal{B}_s\mathcal{B}_q(n-1)\, C_{jkm} y^q - 2 b_{ls}\mathcal{B}_n\mathcal{B}_q(n-1)\, C_{jkm} y^q - 2\, b_l\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q(n-1)\, C_{jkm} y^q - \frac{1}{3}\, a_{lmns}\left(R_{jk} - g_{jk}\, R\right) - \frac{1}{3}\mathcal{B}_s\mathcal{B}_n\, \mathcal{B}_m\mathcal{B}_l\left(R_{jk} - g_{jk}\, R\right) \qquad (32)$$

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, H_k = a_{lmns}\, H_k + b_{lmns}\,(n-1)\, y_k - \frac{1}{3}\, a_{lmns}\,(H_k - y_k R) - \frac{1}{3}\, \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l(H_k - y_k\, R) \quad . \qquad (33)$$

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, H = a_{lmns}\, H + b_{lmns}\, F^2 - \frac{1}{3}\, a_{lmns}\,((n-1)H - F^2 R) - \frac{1}{3}\mathcal{B}_s\mathcal{B}_n\, \mathcal{B}_m\mathcal{B}_l\,((n-1)H - F^2 R) \qquad (34)$$

This following is derived

Theorem 3.4. In $G\mathcal{B}W\text{-}FRF_n$, the R-Ricci tensor R_{jk}, the curvature vector H_k and the scalar curvature H are necessarily non-vanishing.

Compound Derivations

It is known that Cartan's third curvature tensor R^i_{jkh} and Cartan's fourth curvature tensor K^i_{jkh} are connected by the formula

$$R^i_{jkh} = K^i_{jkh} + C^i_{jr}\, H^r_{hk} \qquad (35)$$

Using (35) in (23), we get

$$W^i_{jkh} = K^i_{jkh} + C^i_{jr} H^r_{hk} + \frac{1}{3}\left(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h\right) \qquad (36)$$

Taking covariant derivative of fourth order (Berwald's covariant derivative) of (36), with respect to x^l, x^m, x^n and x^s, successively, we get

$$\begin{aligned}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W^i_{jkh} &= \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l K^i_{jkh} + \left(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l C^i_{jr}\right)H^r_{hk} + \left(\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l C^i_{jr}\right)(\mathcal{B}_s H^r_{hk})\\ &+ \left(\mathcal{B}_s\mathcal{B}_m\mathcal{B}_l C^i_{jr}\right)(\mathcal{B}_n H^r_{hk}) + \left(\mathcal{B}_m\mathcal{B}_l C^i_{jr}\right)(\mathcal{B}_s\mathcal{B}_n H^r_{hk}) + \left(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_l C^i_{jr}\right)(\mathcal{B}_m H^r_{hk})\\ &+ \left(\mathcal{B}_n\mathcal{B}_l C^i_{jr}\right)(\mathcal{B}_s\mathcal{B}_m H^r_{hk}) + \left(\mathcal{B}_s\mathcal{B}_l C^i_{jr}\right)(\mathcal{B}_n\mathcal{B}_m H^r_{hk}) + \left(\mathcal{B}_l C^i_{jr}\right)(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m H^r_{hk})\\ &+ \left(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m C^i_{jr}\right)(\mathcal{B}_l H^r_{hk}) + \left(\mathcal{B}_n\mathcal{B}_m C^i_{jr}\right)(\mathcal{B}_s\mathcal{B}_l H^r_{hk}) + \left(\mathcal{B}_s\mathcal{B}_m C^i_{jr}\right)(\mathcal{B}_n\mathcal{B}_l H^r_{hk})\\ &+ \left(\mathcal{B}_m C^i_{jr}\right)(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_l H^r_{hk}) + \left(\mathcal{B}_s\mathcal{B}_n C^i_{jr}\right)(\mathcal{B}_m\mathcal{B}_l H^r_{hk}) + \left(\mathcal{B}_n C^i_{jr}\right)(\mathcal{B}_s\mathcal{B}_m\mathcal{B}_l H^r_{hk})\\ &+ \left(\mathcal{B}_s C^i_{jr}\right)(\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l H^r_{hk}) + \left(C^i_{jr}\right)(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l H^r_{hk}) + \frac{1}{3}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)\end{aligned} \qquad (37)$$

Using the condition (18) and (36) in (37), we get

$$\begin{aligned}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, K^i_{jkh} &= a_{lmns}\, K^i_{jkh} + a_{lmns}\, C^i_{jr} H^r_{hk} + \frac{1}{3}\, a_{lmns}\left(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h\right)\\ &+ b_{lmns}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) - 2 b_{lmn}\, \mathcal{B}_q\left(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\right) y^q - 2\, b_{lms}\mathcal{B}_q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) y^q\\ &- 2\, b_{lm}\, \mathcal{B}_s\mathcal{B}_q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) y^q - 2\, b_{lns}\, \mathcal{B}_p\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) y^q\\ &- 2\, b_{ln}\mathcal{B}_s\mathcal{B}_q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) y^q - 2 b_{ls}\mathcal{B}_n\mathcal{B}_q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) y^q\\ &- 2\, b_l\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) y^q - \left(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l C^i_{jr}\right)H^r_{hk} - \left(\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l C^i_{jr}\right)(\mathcal{B}_s H^r_{hk})\\ &- \left(\mathcal{B}_s\mathcal{B}_m\mathcal{B}_l C^i_{jr}\right)(\mathcal{B}_n H^r_{hk}) - \left(\mathcal{B}_m\mathcal{B}_l C^i_{jr}\right)(\mathcal{B}_s\mathcal{B}_n H^r_{hk}) - \left(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_l C^i_{jr}\right)(\mathcal{B}_m H^r_{hk})\\ &- \left(\mathcal{B}_n\mathcal{B}_l C^i_{jr}\right)(\mathcal{B}_s\mathcal{B}_m H^r_{hk}) - \left(\mathcal{B}_s\mathcal{B}_l C^i_{jr}\right)(\mathcal{B}_n\mathcal{B}_m H^r_{hk}) - \left(\mathcal{B}_l C^i_{jr}\right)(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m H^r_{hk})\\ &- \left(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m C^i_{jr}\right)(\mathcal{B}_l H^r_{hk}) - \left(\mathcal{B}_n\mathcal{B}_m C^i_{jr}\right)(\mathcal{B}_s\mathcal{B}_l H^r_{hk}) - \left(\mathcal{B}_s\mathcal{B}_m C^i_{jr}\right)(\mathcal{B}_n\mathcal{B}_l H^r_{hk})\\ &- \left(\mathcal{B}_m C^i_{jr}\right)(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_l H^r_{hk}) - \left(\mathcal{B}_s\mathcal{B}_n C^i_{jr}\right)(\mathcal{B}_m\mathcal{B}_l H^r_{hk}) - \left(\mathcal{B}_n C^i_{jr}\right)(\mathcal{B}_s\mathcal{B}_m\mathcal{B}_l H^r_{hk})\\ &- \left(\mathcal{B}_s C^i_{jr}\right)(\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l H^r_{hk}) - C^i_{jr}(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l H^r_{hk}) - \frac{1}{3}\, \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) \quad .\end{aligned} \qquad (38)$$

This shows that

$$\begin{aligned}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, K^i_{jkh} &= a_{lmns}\, K^i_{jkh} + b_{lmns}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) - 2 b_{lmn}\, \mathcal{B}_q\left(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\right) y^q\\ &- 2\, b_{lms}\mathcal{B}_q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) y^q - 2\, b_{lm}\, \mathcal{B}_s\mathcal{B}_q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) y^q\\ &- 2\, b_{lns}\, \mathcal{B}_p\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) y^q - 2\, b_{ln}\mathcal{B}_s\mathcal{B}_q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) y^q\\ &- 2 b_{ls}\mathcal{B}_n\mathcal{B}_q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) y^q - 2\, b_l\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) y^q \,.\end{aligned}$$

If and only if

$$\begin{aligned}&a_{lmns} C^i_{jr} H^r_{hk} + \frac{1}{3} a_{lmns}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) + \left(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l C^i_{jr}\right)H^r_{hk} + \left(\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l C^i_{jr}\right)(\mathcal{B}_s H^r_{hk})\\ &+ \left(\mathcal{B}_s\mathcal{B}_m\mathcal{B}_l C^i_{jr}\right)(\mathcal{B}_n H^r_{hk}) + \left(\mathcal{B}_m\mathcal{B}_l C^i_{jr}\right)(\mathcal{B}_s\mathcal{B}_n H^r_{hk}) + \left(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_l C^i_{jr}\right)(\mathcal{B}_m H^r_{hk})\\ &+ \left(\mathcal{B}_n\mathcal{B}_l C^i_{jr}\right)(\mathcal{B}_s\mathcal{B}_m H^r_{hk}) + \left(\mathcal{B}_s\mathcal{B}_l C^i_{jr}\right)(\mathcal{B}_n\mathcal{B}_m H^r_{hk}) + \left(\mathcal{B}_l C^i_{jr}\right)(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m H^r_{hk})\\ &+ \left(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m C^i_{jr}\right)(\mathcal{B}_l H^r_{hk}) + \left(\mathcal{B}_n\mathcal{B}_m C^i_{jr}\right)(\mathcal{B}_s\mathcal{B}_l H^r_{hk}) + \left(\mathcal{B}_s\mathcal{B}_m C^i_{jr}\right)(\mathcal{B}_n\mathcal{B}_l H^r_{hk})\\ &+ \left(\mathcal{B}_m C^i_{jr}\right)(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_l H^r_{hk}) + \left(\mathcal{B}_s\mathcal{B}_n C^i_{jr}\right)(\mathcal{B}_m\mathcal{B}_l H^r_{hk}) + \left(\mathcal{B}_n C^i_{jr}\right)(\mathcal{B}_s\mathcal{B}_m\mathcal{B}_l H^r_{hk})\\ &+ \left(\mathcal{B}_s C^i_{jr}\right)(\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l H^r_{hk}) + C^i_{jr}(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l H^r_{hk}) + \frac{1}{3}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) = 0\end{aligned} \qquad (39)$$

This following is derived

Theorem 4.1. In $G\mathcal{B}W\text{-}FRF_n$, Cartan's fourth curvature tensor K^i_{jkh} is generalized fourth recurrent if and only if the condition (39) holds good.

Contracting the indices i and h in the conditions (38), using (2), (13h), (13e), (17a), we get

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l K_{jk} = a_{lmns}\, K_{jk} + a_{lmns}\, C^i_{jr} H^r_{ik} + \frac{1}{3} a_{lmns}\big(R_{jk} - g_{jk}R\big) + b_{lmns}(n-1)\, g_{jk}$$
$$-2b_{lmn}\mathcal{B}_q\big(nC_{jks} - C_{jks}\big)y^q - 2b_{lms}\mathcal{B}_q\big(nC_{jkn} - C_{jkn}\big)y^q - 2b_{lm}\,\mathcal{B}_s\mathcal{B}_q\big(nC_{jkn} - C_{jkn}\big)\, y^q$$
$$-2b_{lns}\,\mathcal{B}_p\big(nC_{jkm} - C_{jkm}\big)y^q - 2b_{ln}\mathcal{B}_s\mathcal{B}_q\big(nC_{jkm} - C_{jkm}\big)y^q - 2b_{ls}\mathcal{B}_n\mathcal{B}_q\big(nC_{jkm} - C_{jkm}\big)\, y^q$$
$$-2\, b_l\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q\big(nC_{jkm} - C_{jkm}\big)y^q - \big(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l C^i_{jr}\big)H^r_{ik} - \big(\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l C^i_{jr}\big)\big(\mathcal{B}_s H^r_{ik}\big)$$
$$-\big(\mathcal{B}_s\mathcal{B}_m\mathcal{B}_l C^i_{jr}\big)\big(\mathcal{B}_n H^r_{ik}\big) - \big(\mathcal{B}_m\mathcal{B}_l C^i_{jr}\big)\big(\mathcal{B}_s\mathcal{B}_n H^r_{ik}\big) - \big(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_l C^i_{jr}\big)\big(\mathcal{B}_m H^r_{ik}\big)$$
$$-\big(\mathcal{B}_n\mathcal{B}_l C^i_{jr}\big)\big(\mathcal{B}_s\mathcal{B}_m H^r_{ik}\big) - \big(\mathcal{B}_s\mathcal{B}_l C^i_{jr}\big)\big(\mathcal{B}_n\mathcal{B}_m H^r_{ik}\big) - \big(\mathcal{B}_l C^i_{jr}\big)\big(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m H^r_{ik}\big)$$
$$-\big(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m C^i_{jr}\big)\big(\mathcal{B}_l H^r_{ik}\big) - \big(\mathcal{B}_n\mathcal{B}_m C^i_{jr}\big)\big(\mathcal{B}_s\mathcal{B}_l H^r_{ik}\big) - \big(\mathcal{B}_s\mathcal{B}_m C^i_{jr}\big)\big(\mathcal{B}_n\mathcal{B}_l H^r_{ik}\big)$$
$$-\big(\mathcal{B}_m C^i_{jr}\big)\big(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_l H^r_{ik}\big) - \big(\mathcal{B}_s\mathcal{B}_n C^i_{jr}\big)\big(\mathcal{B}_m\mathcal{B}_l H^r_{ik}\big) - \big(\mathcal{B}_n C^i_{jr}\big)\big(\mathcal{B}_s\mathcal{B}_m\mathcal{B}_l H^r_{ik}\big)$$
$$-\big(\mathcal{B}_s C^i_{jr}\big)\big(\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l H^r_{ik}\big) - C^i_{jr}\big(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l H^r_{ik}\big) - \tfrac{1}{3}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\big(\, R_{jk} - g_{jk}R\big)$$

This shows that

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l K_{jk} = a_{lmns}\, K_{jk} + (n-1)b_{lmns} g_{jk} - 2b_{lmn}\,\mathcal{B}_q\big(n\,C_{jks} - C_{jks}\big)\, y^q \tag{40}$$
$$-2b_{lms}\mathcal{B}_q\big(nC_{jkn} - C_{jkn}\big)y^q - 2b_{lm}\,\mathcal{B}_s\mathcal{B}_q\big(nC_{jkn} - C_{jkn}\big)y^q - 2b_{lns}\,\mathcal{B}_p\big(nC_{jkm} - C_{jkm}\big)\, y^q$$
$$-2b_{ln}\mathcal{B}_s\mathcal{B}_q\big(nC_{jkm} - C_{jkm}\big)y^q - 2b_{ls}\mathcal{B}_n\mathcal{B}_q\big(nC_{jkm} - C_{jkm}\big)y^q - 2b_l\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q\big(nC_{jkm} - C_{jkm}\big)y^q.$$

If and only if

$$a_{lmns}\, C^i_{jr} H^r_{ik} + \tfrac{1}{3} a_{lmns}\,\big(R_{jk} - g_{jk}R\big) - \big(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l C^i_{jr}\big)H^r_{ik} - \big(\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l C^i_{jr}\big)\big(\mathcal{B}_s H^r_{ik}\big)$$
$$-\big(\mathcal{B}_s\mathcal{B}_m\mathcal{B}_l C^i_{jr}\big)\big(\mathcal{B}_n H^r_{ik}\big) - \big(\mathcal{B}_m\mathcal{B}_l C^i_{jr}\big)\big(\mathcal{B}_s\mathcal{B}_n H^r_{ik}\big) - \big(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_l C^i_{jr}\big)\big(\mathcal{B}_m H^r_{ik}\big)$$
$$-\big(\mathcal{B}_n\mathcal{B}_l C^i_{jr}\big)\big(\mathcal{B}_s\mathcal{B}_m H^r_{ik}\big) - \big(\mathcal{B}_s\mathcal{B}_l C^i_{jr}\big)\big(\mathcal{B}_n\mathcal{B}_m H^r_{ik}\big) - \big(\mathcal{B}_l C^i_{jr}\big)\big(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m H^r_{ik}\big)$$
$$-\big(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m C^i_{jr}\big)\big(\mathcal{B}_l H^r_{ik}\big) - \big(\mathcal{B}_n\mathcal{B}_m C^i_{jr}\big)\big(\mathcal{B}_s\mathcal{B}_l H^r_{ik}\big) - \big(\mathcal{B}_s\mathcal{B}_m C^i_{jr}\big)\big(\mathcal{B}_n\mathcal{B}_l H^r_{ik}\big)$$
$$-\big(\mathcal{B}_m C^i_{jr}\big)\big(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_l H^r_{ik}\big) - \big(\mathcal{B}_s\mathcal{B}_n C^i_{jr}\big)\big(\mathcal{B}_m\mathcal{B}_l H^r_{ik}\big) - \big(\mathcal{B}_n C^i_{jr}\big)\big(\mathcal{B}_s\mathcal{B}_m\mathcal{B}_l H^r_{ik}\big)$$
$$-\big(\mathcal{B}_s C^i_{jr}\big)\big(\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l H^r_{ik}\big) - C^i_{jr}\big(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l H^r_{ik}\big) - \tfrac{1}{3}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\big(\, R_{jk} - g_{jk}R\big) = 0$$

This following is derived

Theorem 4.2. In G$\mathcal{B}$W-FRF$_n$, Berwald covariant derivative of the fourth orders for the K-Ricci tensor K_{jk} is generalized fourth recurrent if and only if the condition (40) hold good.

It is known that Cartan's third curvature tensor R^i_{jkh} and Cartan's 2nd curvature tensor P^i_{jkh} are connected by the formula

$$R^i_{jkh} = P^i_{jkh} + \frac{1}{3}\big(\delta^i_h\, R_{jk} - \delta^i_k\, R_{jh}\big) \tag{41}$$

Using (3.1) in (4.7), we get

$$W^i_{jkh} = P^i_{jkh} + \frac{1}{3}\big(\delta^i_h\, R_{jk} - \delta^i_k\, R_{jh}\big) + \frac{1}{3}\big(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h\,\big)$$

Taking covariant derivative of fourth order (Berwald's covariant derivative) of above equation, with respect to x^l, x^m, x^n and x^s, successively, we get

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W^i_{jkh} = \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, P^i_{jkh} + \tfrac{1}{3}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,\big(\delta^i_h\, R_{jk} - \delta^i_k\, R_{jh}\big)$$
$$+\tfrac{1}{3}\,\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,\big(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h\big)\quad . \tag{42}$$

Using the condition (18) in (42), we get

$$\mathcal{B}_s\mathcal{B}_n\,\mathcal{B}_m\mathcal{B}_l\, P^i_{jkh} = a_{lmns} P^i_{jkh} + b_{lmns}\big(\delta^i_h g_{jk} - \delta^i_k g_{jh}\big) - 2b_{lmn}\,\mathcal{B}_q\big(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\big)\, y^q$$
$$-2\, b_{lms}\mathcal{B}_q\big(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\big)\, y^q - 2b_{lm}\,\mathcal{B}_s\mathcal{B}_q\big(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\big)\, y^q$$
$$-2\, b_{lns}\,\mathcal{B}_p\big(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\big)\, y^q - 2b_{ln}\mathcal{B}_s\mathcal{B}_q\big(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\big)\, y^q$$
$$-2\, b_{ls}\mathcal{B}_n\mathcal{B}_q\big(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\big)\, y^q - 2b_l\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q\big(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\big)\, y^q \tag{43}$$
$$-\tfrac{1}{3}\, a_{lmns}\,\big(\delta^i_h R_{jk} - \delta^i_k\, R_{jh}\big) - \tfrac{1}{3} a_{lmns}\,\big(\delta^i_k R_{jh} - g_{jk}\, R^i_h\big) - \tfrac{1}{3}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\big(\delta^i_h\, R_{jk} - \delta^i_k R_{jh}\big)$$
$$+\tfrac{1}{3}\,\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\big(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h\big)\quad .$$

This shows that

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, P^i_{jkh} = a_{lmns}\, P^i_{jkh} + b_{lmns}\big(\delta^i_h\, g_{jk} - \delta^i_k g_{jh}\big) - 2b_{lmn}\,\mathcal{B}_q\big(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\big)\, y^q$$
$$-2\, b_{lms}\mathcal{B}_q\big(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\big)\, y^q - 2\, b_{lm}\,\mathcal{B}_s\mathcal{B}_q\big(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\big)\, y^q$$
$$-2\, b_{lns}\,\mathcal{B}_p\big(\,\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\big)\, y^q - 2\, b_{ln}\mathcal{B}_s\mathcal{B}_q\big(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\big)\, y^q$$
$$-2\, b_{ls}\mathcal{B}_n\mathcal{B}_q\big(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\big)\, y^q - 2\, b_l\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q\big(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\big)\, y^q\quad .$$

If and only if

$$a_{lmns}\big(\delta^i_h\, R_{jk} - \delta^i_k\, R_{jh}\big) - a_{lmns}\big(\delta^i_k R_{jh} - g_{jk} R^i_h\big) - \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\big(\delta^i_h R_{jk} - \delta^i_k R_{jh}\big)$$
$$+\,\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\big(\delta^i_k R_{jh} - g_{jk}\, R^i_h\big) = 0\quad . \tag{44}$$

This following is derived

Theorem 4.3. In G$\mathcal{B}$W - FRF_n ,Cartan's 2nd curvature tensor P^i_{jkh} is generalized fourth recurrent if and only if the condition (44) holds good.

Transvecting (43) by y^j, using (1a), (4b), (1a), (12a), and (13c), we get

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, P^i_{kh} = a_{lmns}\, P^i_{kh} + b_{lmns}\big(\delta^i_h\, y_k - \delta^i_k\, y_h\big) - \tfrac{1}{3}\, a_{lmns}\, \big(\delta^i_h H_k - \delta^i_k H_h\big)$$
$$- \tfrac{1}{3} a_{lmns}\, \big(\delta^i_k H_h - y_k R^i_h\big) - \tfrac{1}{3}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\big(\delta^i_h H_k - \delta^i_k H_h\big) - \tfrac{1}{3}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\big(\delta^i_k H_h - y_k R^i_h\big) \tag{45}$$

This shows that

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, P^i_{kh} = a_{lmns}\, P^i_{kh} + b_{lmns}\big(\delta^i_h y_k - \delta^i_k y_h\big)$$

If and only if

$$a_{lmns}\, \big(\delta^i_h H_{jk} - \delta^i_k H_h\big) - a_{lmns}\big(\delta^i_k H_h - y_k R^i_h\big) - \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\big(\delta^i_h H_{jk} - \delta^i_k H_h\big)$$
$$+ \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\big(\delta^i_k H_h - y R^i_h\big) = 0 \quad . \tag{46}$$

This following is derived

Theorem 4.4. In $G\mathcal{B}W\text{-}FRF_n$,Cartan's 2nd curvature tensor is the four orders for the torsion tensor P^i_{jk} is generalized fourth recurrent if and only if the condition (46) hold good.

Contracting the indices i and h in the conditions (45), using (2), (12e) and (13h), we get

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, P_k = a_{lmns}\, P_k + (n-1)\, b_{lmns}\, y_k - \tfrac{1}{3}(n-1)\, a_{lmns}\, H_k$$
$$- \tfrac{1}{3} a_{lmns}\, (H_k - y_k R) - \tfrac{1}{3}(n-1)\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l H_k - \tfrac{1}{3}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l(H_k - y_k R).$$

This shows that

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, P_k = a_{lmns}\, P_k + (n-1)\, b_{lmns}\, y_k$$

If and only if

$$a_{lmns} H_k + a_{lmns}(H_k - y_k R) + \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l H_k + \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l(H_k - y_k R) = 0 \quad . \tag{47}$$

This following is derived

Theorem 4.5. In $G\mathcal{B}W\text{-}FRF_n$,the Cartan's second curvature tensor of fourth order for the torsion tensor is generalized fourth recurrent if and only if condition (47) holds.

The addition of an obstacle barrier guarantees that the system never encounters a circular obstruction, and the addition of a barrier geometry bends those geodesic pathways away from the walls. The geodesics then randomly twist and bend when a random vortex geometry is applied.

Because of the contributions from the earlier layers, the final geodesics never pass through either kind of barrier despite the random movements. Lastly, the geodesics are heuristically directed toward a destination via an attractor geometry. Every geometry's priority in the weighted average is shaped by Finsler energies, so it only takes center stage when it is most crucial. See (Ratliff, N.D. et al., 2020) for a thorough description of the geometries and Finsler energies used in this case. According to the theory of geometric textiles, geometries constructed in this way may be optimized to offer system objectives represented by local minima of an objective potential by simply pressing the system with the potential's negative gradient. We use a straightforward attractor potential of the kind outlined in (Ratliff, N.D et al., 2020) to do this for each of these geometries. The geometry significantly influences the system's behavior along the way in each scenario, but the theory of textiles ensures that all differential equations eventually converge to the aim.

Conclusions and Recommendations

A Finsler space meets the criterion (18), it is referred to as a generalized BW-fourth recurrent space. The B-covariant derivative of the four orders for Wely's projective deviation tensor W^i_h and Wely's projective torsion tensorW^i_{kh}i in $G\mathcal{B}W\text{-}FRF_n$ (19) and (24), respectively, provide i. Cartan's third curvature tensor R^i_{jkh} is generalized fourth recurrent in $G\mathcal{B}W\text{-}\mathrm{FRF}_n$ if and only if equation (27) equals zero. Similarly, the h(v) –torsion tensor H^i_{kh} is generalized fourth recurrent if and only if equation (29) equals zero. In $G\mathcal{B}W\text{-}\mathrm{FRF}_n$, the deviation tensor H^i_h is generalized fourth recurrent if and only if the equation (29) is equal to zero. In $G\mathcal{B}W\text{-}\mathrm{FRF}_n$ Cartan's fourth curvature tensor K^i_{jkh} is generalized fourth-recurrent if and only if the condition (39) holds good. If and only if the condition (40) is met, the Berwald covariant derivative of the four orders for the K-Ricci tensor K_{jk} is generalized fourth recurrent. The authors discuss the features of special spaces for Finsler space and suggest that further study and development in generalized $\mathcal{B}W$- five recurrent Finsler spaces is necessary.

Future Applications in Robotics

The results obtained in this study lay a foundational framework for potential applications of higher-order recurrent Finsler structures particularly the Generalized BW-Four-Recurrent Finsler spaces $G\mathcal{B}W\text{-}FRF_n$ in the field of intelligent robotics. As robotic systems become increasingly autonomous and are deployed in dynamically complex, anisotropic, and non-Euclidean environments, there arises a growing need for advanced geometric models capable of capturing directional dependencies and curvature-induced behaviors in a precise and adaptable manner.

The recurrence relations discovered in this study, including fourth th-order Berwald covariant derivatives of curvature tensors, give a mathematically complex structure that may be used to describe and regulate the behavior of robots operating in such situations. Specifically, $G\mathcal{B}W\text{-}FRF_n$ spaces offer a theoretical basis for describing robot kinematics and dynamics in terrains where classic Riemannian or Euclidean models fall short:

1. Path planning and navigation algorithms that leverage the anisotropic curvature of the environment to maximize movement efficiency.
2. Adaptive control systems for mobile robots rely on real-time geometric feedback to make decisions.
3. Sensor-based mapping uses curvature and directional deviation tensors to help robots comprehend their surroundings.
4. In swarm robotics, interactions between several agents may be described using recurrent geometric relations in non-flat areas.

These possibilities indicate that Finsler geometry, particularly in its generalized recurrent versions, can play an important role in the creation of next-generation robotic systems. Future interdisciplinary research between differential geometers and robotics engineers is essential to translate the theoretical advancements presented here into practical algorithms and hardware implementations.

Author contributions: Al-Qashbari and Abdullah: Conceptualization; investigation, validation, writing-original draft preparation; **Ahmed and Irhouma**: Review and editing, and formal analysis. All authors have read and agreed to the published version of the manuscript.

Data availability statement: "The datasets generated during the current study are not publicly available as the study is on-going; however, they are available from the corresponding author upon reasonable request."

Declaration of conflict of interest: "The authors confirm that they have no conflicts of interest to declare."

Funding information: "No funding is received for this work."

References

[1] A. Abdallah, A. Navlekar, and K. Ghadle. "On special spaces of torsion tensor in generalized recurrent Finsler space." *Bulletin of Pure & Applied Sciences Mathematics and Statistics*, vol. 41, no. 1, pp. 74-80, 2022. https://doi.org/10.5958/2320-3226.2022.00011.X

[2] A. Abdallah, A. Navlekar, and A. Hamoud. "On birecurrent for some tensors in various Finsler spaces." *Journal of Finsler Geometry and Its Applications*, vol. 4, pp. 33-44, 2023. https://doi.org/10.22098/jfga.2023.12052.1083

[3] A. Al-Qashbari. "Recurrence decompositions in Finsler space". *Journal of Mathematical Analysis and Modeling*, vol. 1, no. 1, pp. 77-86, 2020. https://doi.org/10.48185/jmam.v1i1.40

[4] A. Al-Qashbari. "Finsler space of fourth order on some properties of Rh-generalized recurrent." *Journal of Science and Technology*, vol. 28, no. 2, pp. 14-22, 2023. https://doi.org/10.20428/jst.v28i2.2034

[5] A. Al-Qashbari, and F. Al-Sallal. "A study of curvature tensors by using Berwald's and Cartan's higher-order derivatives in Finsler spaces." *Technological Applied and Humanitarian Academic Journal*, vol. 1, no. 1, pp. 11-27, 2024.

[6] A. Al-Qashbari, and F. Al-Sallal. "Decomposition of generalized recurrent tensor fields of Rh-nth order in Finsler manifolds". *Journal of Science and Technology*, vol. 30, no. 2, pp. 11-27, 2025. https://doi.org/10.20428/jst.v30i2.2664

[7] A. Al-Qashbari, and A. Mubark. "A study of recurrent Finsler spaces of higher order with Cartan's curvature tensor." *University of Aden Journal of Natural and Applied Sciences*, vol. 27, no. 2, pp. 291-300, 2023. https://doi.org/10.47372/uajnas.2023.n2.a10

[8] A. Al-Qashbari. "A study of the M-projective curvature tensor in generalized recurrent and birecurrent Finsler spaces." *Journal of Science and Technology*, vol. 30, no. 6, pp. 16-25, 2025. https://doi.org/10.20428/jst.v30i6.2917

[9] A. Al-Qashbari, B. Muthanna, W. Hadi, F. AL-ssallal, C. Arroussi, I. Chekalil, and A. Mandal. "Higher-Order Recurrent Finsler Geometry and Anisotropic Flow Modelling with CFD Applications." *Wadi Alshatti University Journal of Pure and Applied Sciences*, vol. 4, no. 2, pp. 323-330, 2026. https://doi.org/10.63318/waujpasv4i2-37

[10] S. AbuShanab, and A. Hadoush. "Hardware Implementation of Newton-Raphson Method for Nonlinear Stress Estimation." *Wadi Alshatti University Journal of Pure and Applied Sciences*, vol. 3, no. 2, pp. 244-250, 2025. https://doi.org/10.63318/waujpasv3i2-30

[11] A. Hadoush. "Internally Balanced Spatial Elasticity Tensor." *Wadi Alshatti University Journal of Pure and Applied Sciences*, vol. 3, no. 2, pp. 98-101, 2025. https://doi.org/10.63318/waujpasv3i2-11

[12] A. Al-Qashbari. and S. Baleedi. "Study on Generalized BK-5th Recurrent Finsler Space." *Computational Mathematics and its Applications*, vol. 1, no. 1, pp. 1-14, 2024. https://doi.org/10.17352/cma.000002

[13] A. Habeb. and A. Alashlam. "Numerical Solution of Natural Convection in Vertical Channels." *Wadi Alshatti University Journal of Pure and Applied Sciences*, vol. 3, no. 2, pp. 138-148, 2025. https://doi.org/10.63318/waujpasv3i2-18

[14] C. Li. "Curvatures and hyperbolic flows for natural mechanical systems in Finsler geometry". *Journal of Dynamical and Control Systems*, vol. 27, 2021. https://doi.org/10.1007/s10883-021-09533-6

[15] N. Cui, J. Guo, and L. Zhou. "A uniformization theorem in complex Finsler geometry." 2021. https://doi.org/10.48550/arXiv.2102.13484

[16] C. Li, H. Koibuchi, and V. Egorov. "Nonlinear spectrums of Finsler manifolds." *Mathematische Zeitschrift*, vol. 300, pp. 81-123, 2021. https://doi.org/10.1007/s00209-021-02767-x

[17] S. Ohta. *Comparison Finsler geometry*. Springer, 2021. https://doi.org/10.1007/978-3-030-80650-7

[18] C. Pfeifer. "Finsler spacetime geometry in physics." *International Journal of Geometric Methods in Modern Physics*, vol. 16, Suppl. 02, Article 1941004, 2019. https://doi.org/10.1142/S0219887819410044

[19] N. Ratliff, K. Van Wyk, M. Xie, A. Li, and M. A. Rana. "Generalized nonlinear and Finsler geometry for robotics." 2020. https://doi.org/10.1109/ICRA48506.2021.9561543.

[20] H. Rund. *The differential geometry of Finsler spaces*. Springer-Verlag, 1959.

[21] A. Soleiman. *On hyper-generalized recurrent Finsler spaces*. 2019. https://doi.org/10.48550/arXiv.1707.04740.

[22] A. Tayebi, T. Tabatabaeifar, and B. Najafi. "Weighted projective Ricci curvature in Finsler geometry." *Mathematica Slovaca*, vol. 71, no. 1, pp. 183-198, 2021. https://doi.org/10.1515/ms-2017-0446

[23] A. Tomasiello. "Higher spins and Finsler geometry." *Journal of High Energy Physics*, vol. 2024, Article 47, 2024. https://doi.org/10.1007/JHEP10(2024)047.

[24] M. Xu. "Geodesic orbit spheres and constant curvature in Finsler geometry." *Differential Geometry and its Applications*, vol. 61, pp. 197-206, 2018. https://doi.org/10.1016/j.difgeo.2018.07.002

[25] B. Yashwanth, S. Murthy, and Z. Nekouee. "Generalized Finslerian wormhole models in f(R,T) gravity." *Particles*, vol. 7, no. 3, pp. 747-767, 2023. https://doi.org/10.3390/particles7030043

[26] J. Zhu and B.-Q. Ma. "Lorentz violation in Finsler geometry." *Symmetry*, vol. 15, no. 5, Article 978, 2023. https://doi.org/10.3390/sym15050978.

[27] N. Fathi. "Analytical-numerical computation of view factor for several arrangements of two rectangular surfaces with non-common edge." *International journal of heat and mass transfer*, vol. 159, p. 120130, 2020. https://doi.org/10.1016/j.ijheatmasstransfer.2020.120130

[28] S. Alsadi, et al. "View factors in horizontal plane fixed-mode solar PV fields." *Frontiers in Energy Research*, vol. 10, p. 859075, 2023. https://doi.org/10.3389/fenrg.2022.859075

[29] S. Bashier, "Geometry of Euclidean Spaces under Generalized Topology." *Wadi Alshatti University Journal of Pure and Applied Sciences*, vol. 4, no. 2, pp. 368-373, 2026. https://doi.org/10.63318/waujpasv4i2-42

[30] F. Ahmed, M. Mansour, and M. El-Horbaty. "Analytical Solution and Numerical Simulation of the Two-Dimensional Time-Fractional Reaction-Diffusion Brusselator System." *Wadi Alshatti University Journal of Pure and Applied Sciences*, vol. 4, no. 2, pp. 351-358, 2026. https://doi.org/10.63318/waujpasv4i2_40