

A (p, q) -Analogue of the Two-Variable Humbert Function: Generating Functions, Difference Equations, and Recurrence Relations for Engineering Computation

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ABSTRACT

This paper introduces a (p, q) -analogue of the two-variable Humbert function within post-quantum calculus. Using generating functions and (p, q) -differential operators, an explicit representation is derived, and its generating function, difference equations, recurrence relations, and polynomial identities are established. These results extend selected properties of classical Humbert functions to the (p, q) -setting and provide a theoretical foundation for future symbolic and numerical applications in engineering and applied mathematics.

نظير (p, q) لدالة هومبرت ذات المتغيرين: الدوال المؤلفة، ومعادلات الفروق، والعلاقات التراجعية من أجل الحسابات الهندسية

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المخلص	الكلمات المفتاحية
تقدّم هذه الورقة نظرياً من نوع (p, q) لدالة هومبرت ذات المتغيرين في إطار حساب التفاضل والتكامل ما بعد الكمي. وباستخدام الدوال المؤلفة ومؤثرات التفاضل من نوع (p, q) ، يُشتق تمثيل صريح لهذه الدالة، كما تُحدّد دالتها المؤلفة ومعادلات الفروق والعلاقات التراجعية والهويات متعددة الحدود المرتبطة بها. وتوسّع هذه النتائج بعض خصائص دوال هومبرت الكلاسيكية إلى إطار (p, q) ، كما توفر أساساً نظرياً للتطبيقات الرمزية والعديدية المستقبلية في الهندسة والرياضيات التطبيقية.	حساب التفاضل والتكامل من نوع (p, q) دالة هومبرت ذات المتغيرين الدوال المؤلفة الحسابات الهندسية

Introduction

Quantum calculus, particularly (q) -calculus and (p, q) -calculus, has attracted considerable attention because of its role in generalizing classical differential and integral operators and in developing new families of special functions and polynomials. The foundations of quantum calculus and two-parameter quantum algebras have been investigated by Kac and Cheung [14], Chakrabarti and Jagannathan [6], Jagannathan and Srinivasa Rao [13], and Sahai and Yadav [24]. Further contributions to the theory of quantum-deformed algebras and special functions were presented by Bukweli-Kyemba and Hounkonnou [4], while Burban and Klimyk [5] developed differentiation, integration, and hypergeometric functions associated with two-parameter quantum groups. These studies provide a mathematical foundation for extending classical analytical methods to quantum-calculus settings.

Quantum calculus and the theory of special functions have attracted considerable attention because of their applications in mathematical analysis, polynomial theory, and

approximation theory. The foundations of quantum calculus and its generalizations were discussed by Kac and Cheung [14], while Burban and Klimyk [5] investigated (p, q) -differentiation, integration, and hypergeometric functions related to quantum groups. Further developments in two-parameter quantum algebras and their associated special functions were presented by Chakrabarti and Jagannathan [6], Jagannathan and Srinivasa Rao [13], and Sahai and Yadav [27]. These studies provide an important theoretical framework for extending classical mathematical structures to (q) - and (p, q) -analogues.

Several researchers have subsequently focused on generalized polynomial families and their properties. Acikgoz et al. [1] investigated (q) -analogues of Apostol-type numbers and polynomials, while Duran et al. [7] introduced a unified (q) -analogue of Apostol-type polynomials of order (α) . Related investigations include (q) -Hermite polynomials [8], α -Bernoulli, (p, q) -Euler, and (p, q) -Genocchi polynomials [9], and other polynomial families derived from (q) -calculus [10]. Furthermore, Sadjang [26] studied (q) -Appell

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polynomials, whereas Kızılateş et al. [17] examined (q)-Chebyshev polynomials and related polynomial systems. More recent contributions have extended these investigations to (q)-Hermite-based Appell polynomials in multivariate and three-dimensional settings [34, 35], as well as (q)-Legendre-based Gould Hopper polynomials using operational methods [18].

Another important direction concerns Humbert functions and their generalizations. Varma [33] studied Humbert functions, providing an early contribution to this area. Subsequent research considered new (q)-extensions of these functions [31] and their generating-function representations [28]. In addition, Alsarahi [2] investigated a two-variable analogue of the Humbert function, while Jaimini et al. [11] studied a generalized class of multivariable Humbert-type polynomials. Shehata [29] also derived new formulas for the bibasic Humbert hypergeometric functions (Ψ_{11}) and (Ψ_{12}). These contributions demonstrate continuing interest in extending classical special functions to generalized and multivariable frameworks.

Furthermore, operational methods, integral transforms, and approximation processes have played an important role in the development of generalized function theory. Araci et al. [3] studied a (q)-derivative operator and associated divided differences, while Sadjang [24, 25] investigated (q)-analogues of the Laplace transform and fundamental results of (p,q)-calculus. In approximation theory, Mursaleen et al. [19, 20] examined (q)-Bernstein operators and approximation by (q)-Lorentz polynomials, respectively. Khan and Lobiyal [15] explored applications of (q)-analogues of Bernstein functions to Bézier curves. Other related studies addressed fractional-integral transformations of hypergeometric functions [21, 22] and generalized special functions using umbral methods [16].

Addressed numerical methods for nonlinear problems, computational solutions of differential equations, and tensor-based mathematical modeling in engineering applications [36–38].

Motivated by these developments, further investigation of generalized special functions and polynomial systems remains of interest, particularly in relation to their generating functions, operational representations, and structural properties. The present study is intended to contribute to this broad research area by developing and analyzing the specific mathematical constructions considered herein. Its precise contribution can be established by comparing the proposed definitions and results with the most closely related studies discussed above.

Despite these developments, further investigation is needed into the systematic construction of (p,q)-analogues of multivariable Humbert functions and the derivation of their associated generating functions, difference equations, and recurrence relations within a unified framework. Motivated by the preceding studies, the present paper introduces a (p,q)-analogue of the two-variable Humbert function and investigates its principal analytical and algebraic properties. First, an explicit series representation of the proposed function is established through its defining generating function. Subsequently, generating-function identities, symmetry and parity properties, and limiting relations connecting the new function with its classical and (q)-analogues are derived. Furthermore, differential and recurrence relations are obtained, together with a polynomial

identity that provides additional insight into the structure of the proposed function.

The main contribution of this work is the development of a unified framework for studying these properties in the (p,q)-calculus setting. The results extend selected properties of Humbert-type functions and provide a basis for further analytical and computational investigations of generalized multivariable special functions.

Motivated by these developments, the present study investigates a (p,q)-analogue of the two-variable Humbert function and examines its fundamental analytical and algebraic properties within the framework of post-quantum calculus. The main objective is to derive an explicit representation of the proposed function and establish several associated identities using generating-function techniques and (p,q)-differential operators. In particular, we obtain a generating function, derive corresponding difference equations and recurrence relations, and investigate related polynomial representations and identities.

The rest of this paper is structured as follows:

We first recall the necessary definitions and preliminary results concerning (p,q)-calculus, including the (p,q)-analogue of the Pochhammer symbol, also known as the (p,q)-shifted factorial. We then introduce the proposed two-variable Humbert function and derive its explicit representation. Subsequently, we establish its generating function, associated difference equations, recurrence relations, and related polynomial identities. Finally, we discuss the mathematical significance of these results and their potential relevance to symbolic and numerical methods in engineering computation.

Let us begin by recalling the definition of the (p,q)-analogue of the Pochhammer symbol, or (p,q)-shifted factorial, as given

$$[a]_q = \frac{1-q^a}{1-q}, \quad 0 < |q| < 1; \\ q \in \mathbb{C} - \{1\}; \quad a \in \mathbb{C}. \quad (1)$$

Where

$$\lim_{q \rightarrow 1} [a]_q = \lim_{q \rightarrow 1} \frac{1-q^a}{1-q} = a.$$

Next, we define the q-analogue of n! as

$$[n]_q! = \begin{cases} 1, & n = 0 \\ [n]_q [n-1]_q \dots [2]_q [1]_q, & n \in \mathbb{N}. \end{cases} \quad (2) \text{ Or} \\ [n]_q! = \prod_{k=1}^n [k]_q, \quad [0]_q! = 1, \quad q \in \mathbb{C} \setminus \{0,1\}.$$

The notation $[a]_{p,q}$ represents the (p,q)-number, also known as the bibasic number or twin-basic number.

$$[a]_{p,q} = \frac{p^a - q^a}{p - q}, \quad 0 < |q| < |p| \leq 1; \quad p, q, a \in \mathbb{C}. \quad (3)$$

The (p,q)-number and (p,q)-factorial for $p, q, a \in \mathbb{C}$ and $0 < |q| < |p| \leq 1$ are as follows. Refer to [10,11,19]

$$[n]_{p,q} = \begin{cases} \frac{p^n - q^n}{p - q}, & n \in \mathbb{N} \\ 0, & n = 0 \end{cases}.$$

A logical extension of the q-number is the twin-basic number, which is

$$\lim_{p \rightarrow 1} [n]_{p,q} = [n]_q = \frac{1-q^n}{1-q}, \quad q \neq 1.$$

The definition of the (p,q)-factorial is

$$[n]_{p,q}! = \prod_{k=1}^n [k]_{p,q} = \frac{((p,q):(p,q))_n}{(p-q)^n}, \\ n \geq 1; \quad [0]_{p,q}! = 1. \quad (4)$$

Additionally, let us provide the so-called (p,q)-binomial coefficient.

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}! [k]_{p,q}!}, \quad 0 \leq k \leq n, \quad k, n \in \mathbb{N}. \quad (5)$$

The definition of the (p, q) -exponential function is (see [22,24]).

$$e_{p,q}(x) = \sum_{k=0}^{\infty} p^{\binom{k}{2}} \frac{x^k}{[k]_{p,q}!} . \quad (6)$$

The definition of the (p, q) -complementary exponential function is

$$E_{p,q}(x) = \sum_{k=0}^{\infty} q^{\binom{k}{2}} \frac{x^k}{[k]_{p,q}!} . \quad (7)$$

It is easy to see that

$$e_{p,q}(x) \cdot E_{p,q}(-x) = 1.$$

Let f be a function that is defined on a portion of the real or complex plane. The following is the definition of the function $f(p, q)$ -derivative operator (see [4,19,24]).

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad x \neq 0, \quad (8)$$

and $(D_{p,q}f)(0) = f'(0)$, given that f fulfills the following relations (see [22]) and is differentiable at 0.

$$D_{p,q}e_{p,q}(\mu x) = \mu e_{p,q}(\mu p x) \mu, \quad (9)$$

$$D_{p,q}E_{p,q}(\mu x) = \mu E_{p,q}(\mu q x). \quad (10)$$

It is evident that $D_{p,q}x^n = [n]_{p,q}x^{n-1}$. Additionally, observe that the (p, q) -derivative simplifies to the Hahn derivative provided by for $p = 1$.

$$D_qf(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0.$$

The following product rules are satisfied by the (p, q) -derivative operator (see [22]).

$$D_{p,q}(f(x) \cdot g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x), \quad (11)$$

$$D_{p,q}(f_1(x) \cdot f_2(x) \cdot f_3(x)) = f_3(px) \cdot f_2(px)D_{p,q}\{f_1(x)\} + f_3(px) \cdot f_1(qx)D_{p,q}\{f_2(x)\} + f_1(qx) \cdot f_2(qx)D_{p,q}\{f_3(x)\}. \quad (12)$$

The generating function is used in the previous studies [26] to define the Humbert function $J_{n,m}(x)$.

$$\exp\left(\frac{x}{3}\left(u + t - \frac{1}{ut}\right)\right) = \sum_{m,n=-\infty}^{\infty} J_{n,m}(x) u^m t^n. \quad (13)$$

or, more generally, as follows:

$$J_{n,m}(x) = \left(\frac{x}{3}\right)^{m+n} \frac{1}{\Gamma(m+1)\Gamma(n+1)} {}_0F_2\left(-; m+1, n+1; -\frac{x^3}{27}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(m+k+1)\Gamma(n+k+1)} \left(\frac{x}{3}\right)^{m+n+3k}. \quad (14)$$

A family of novel q -extensions of the Humbert functions of the type was presented by Srivastava and Shehata in [24].

$$J_{n,m}^{(1)}(x/q) = \sum_{k=0}^{\infty} \frac{(-1)^k}{[m+k]_q! [n+k]_q! [k]_q!} \left(\frac{x}{3}\right)^{m+n+3k}. \quad (15)$$

Additionally, Shehata [22] provided an analysis of the (p, q) -Humbert functions from the perspective of the generating function technique described by

$$J_{n,m}^{(1)}(x/p, q) = \sum_{k=0}^{\infty} (-1)^k \frac{p^{\binom{m+k}{2} + \binom{n+k}{2} + \binom{k}{2}}}{[m+k]_{p,q}! [n+k]_{p,q}! [k]_{p,q}!} \left(\frac{x}{3}\right)^{m+n+3k}. \quad (16)$$

Alsarahi [2] introduced and studied of Humbert polynomials $J_{m,n,r,s}(x, y)$ defined by

$$J_{m,n,r,s}(x, y) = \left(\frac{x}{3}\right)^{m+n} \left(\frac{yf(x)}{3}\right)^{r+s} \times \sum_{c,k=0}^{\infty} (-1)^{c+k} \frac{1}{(m+c)!(n+c)!c!(r+k)!(s+k)!k!} \times \left(\frac{x}{3}\right)^{3c} \left(\frac{yf(x)}{3}\right)^{3k}. \quad (17)$$

The (p, q) -analogue Humbert polynomial of two variables is introduced and some of its recurrence relations are examined in the following sections.

The (p, q) -Humbert Function of Two Variables

The generating function that follows defines the (p, q) -Humbert function of two variables, represented by $J_{m,n,r,s}(x, y; p, q)$:

$$\exp_{p,q}\left(\frac{x}{3}\left(u + t - \frac{1}{ut}\right)\right) \cdot \exp_{p,q}\left(\frac{y}{3}\left(v + w - \frac{1}{vw}\right)\right) = \sum_{m,n,r,s=-\infty}^{\infty} J_{m,n,r,s}(x, y; p, q) u^m t^n v^r w^s, \quad (18)$$

where $x, y \in R$, $u, t, v, w \in \mathbb{C}$, $u, t, v, w \neq 0$, $0 < |q| < |p| \leq 1$, $q, p \in \mathbb{C}$.

The (p, q) -Humbert function of two variables $J_{m,n,r,s}(x, y; p, q)$ will now be determined using the generating function mentioned above in the form of the following theorem:

Theorem 2.1.

Assuming that $0 < |q| < |p| \leq 1$, $q, p \in \mathbb{C}$, the formula for the (p, q) -Humbert function of two variables $J_{m,n,r,s}(x, y; p, q)$ is as follows:

$$J_{m,n,r,s}(x, y; p, q) = \left(\frac{x}{3}\right)^{m+n} \left(\frac{y}{3}\right)^{r+s} \times \sum_{c,k=0}^{\infty} (-1)^{c+k} \frac{p^{\binom{m+c}{2} + \binom{n+c}{2} + \binom{c}{2} + \binom{r+k}{2} + \binom{s+k}{2} + \binom{k}{2}}}{[m+c]_{p,q}! [n+c]_{p,q}! [c]_{p,q}! [r+k]_{p,q}! [s+k]_{p,q}! [k]_{p,q}!} \times \left(\frac{x}{3}\right)^{2c} \left(\frac{y}{3}\right)^{2k}. \quad (19)$$

Proof. Let's use W to represent the left side of (18). Then, using (14), we obtain

$$W = e_{p,q}\left[\frac{xu}{3}\right] \cdot e_{p,q}\left[\frac{xt}{3}\right] \cdot e_{p,q}\left[-\frac{x}{3ut}\right] \cdot e_{p,q}\left[\frac{yv}{3}\right] \cdot e_{p,q}\left[\frac{yw}{3}\right] \cdot e_{p,q}\left[-\frac{y}{3vw}\right] = \sum_{a=0}^{\infty} \frac{p^{\binom{a}{2}}}{[a]_{p,q}!} \left(\frac{xu}{3}\right)^a \sum_{b=0}^{\infty} \frac{p^{\binom{b}{2}}}{[b]_{p,q}!} \left(\frac{xt}{3}\right)^b \sum_{c=0}^{\infty} \frac{(-1)^c p^{\binom{c}{2}}}{[c]_{p,q}!} \left(\frac{x}{3ut}\right)^c \times \sum_{i=0}^{\infty} \frac{p^{\binom{i}{2}}}{[i]_{p,q}!} \left(\frac{yv}{3}\right)^i \sum_{j=0}^{\infty} \frac{p^{\binom{j}{2}}}{[j]_{p,q}!} \left(\frac{yw}{3}\right)^j \sum_{k=0}^{\infty} \frac{(-1)^k p^{\binom{k}{2}}}{[k]_{p,q}!} \left(\frac{y}{3vw}\right)^k = \sum_{a,b=0}^{\infty} \sum_{c=0}^{\infty} (-1)^c \frac{p^{\binom{a}{2} + \binom{b}{2} + \binom{c}{2}}}{[a]_{p,q}! [b]_{p,q}! [c]_{p,q}!} \left(\frac{x}{3}\right)^{a+b+c} u^{a-c} t^{b-c} \sum_{i,j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{p^{\binom{i}{2} + \binom{j}{2} + \binom{k}{2}}}{[i]_{p,q}! [j]_{p,q}! [k]_{p,q}!} \left(\frac{y}{3}\right)^{i+j+k} v^{i-k} w^{j-k}. \quad (20)$$

Replace a, b, i and j by $m+c, n+c, r+k$ and $s+k$ respectively in the right-hand side of equation (20), we get

$$W = \sum_{m,n=-\infty}^{\infty} \sum_{c=0}^{\infty} (-1)^c \frac{p^{\binom{m+c}{2} + \binom{n+c}{2} + \binom{c}{2}}}{[m+c]_{p,q}! [n+c]_{p,q}! [c]_{p,q}!} \left(\frac{x}{3}\right)^{m+n+2c} u^m t^n \times \sum_{r,s=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{p^{\binom{r+k}{2} + \binom{s+k}{2} + \binom{k}{2}}}{[r+k]_{p,q}! [s+k]_{p,q}! [k]_{p,q}!} \times \left(\frac{y}{3}\right)^{r+s+2k} v^r w^s. \quad (21)$$

By using relation (11) in (21), we obtain

$$W = \sum_{m,n,r,s=-\infty}^{\infty} \left(\frac{x}{3}\right)^{m+n} \left(\frac{y}{3}\right)^{r+s} \times \sum_{c,k=0}^{\infty} (-1)^{c+k} \frac{p^{\binom{m+c}{2} + \binom{n+c}{2} + \binom{c}{2} + \binom{r+k}{2} + \binom{s+k}{2} + \binom{k}{2}}}{[m+c]_{p,q}! [n+c]_{p,q}! [c]_{p,q}!} \times \frac{1}{[r+k]_{p,q}! [s+k]_{p,q}! [k]_{p,q}!} \left(\frac{x}{3}\right)^{2c} \left(\frac{y}{3}\right)^{2k} u^m t^n v^r w^s. \quad (22)$$

We obtain the connection (19) by equating the coefficients of $u^m t^n v^r w^s$ with the right-hand side of (18).

Remark 2.1: Using equations (14), (15), and (16) and setting $y = 0$ in equation (19), we obtain

$$\lim_{\substack{p \rightarrow 1 \\ q \rightarrow 1}} J_{m,n,r,s}(x, 0; p, q) = J_{m,n}(x), \quad (23)$$

$$\lim_{p \rightarrow 1} J_{m,n,r,s}(x, 0; p, q) = J_{m,n}(x; q), \quad (24)$$

$$J_{m,n,r,s}(x, 0; p, q) = J_{m,n}(x; p, q). \quad (25)$$

respectively.

Remark 2.2.

By relation (4), the series expansions of the (p, q) - Humbert function $J_{m,n,r,s}(x, y; p, q)$ is given as

$$J_{m,n,r,s}(x, y; p, q) = \frac{\left(\frac{x}{3}\right)^{m+n} \left(\frac{y}{3}\right)^{r+s}}{\left((p,q):(p,q)\right)_n \left((p,q):(p,q)\right)_m \left((p,q):(p,q)\right)_r \left((p,q):(p,q)\right)_s} \times \sum_{c=0}^{\infty} \frac{(-1)^c p^{\binom{m+c}{2} + \binom{n+c}{2} + \binom{r+c}{2}}}{\left((p,q):(p,q)\right)_c \left((p^{m+1}, q^{m+1}); (p,q)\right)_c \left((p^{n+1}, q^{n+1}); (p,q)\right)_c} \times \left(\frac{(p-q)x}{3}\right)^{2c} \times \sum_{k=0}^{\infty} \frac{(-1)^k p^{\binom{r+k}{2} + \binom{s+k}{2} + \binom{k}{2}}}{\left((p,q):(p,q)\right)_k \left((p^{r+1}, q^{r+1}); (p,q)\right)_k \left((p^{s+1}, q^{s+1}); (p,q)\right)_k} \times \left(\frac{(p-q)y}{3}\right)^{2k}. \quad (26)$$

Corollary 2.1.

If m, n, r, s be integer, then $J_{m,n,r,s}(x, y; p, q)$ satisfies the following relations:

$$J_{-m,n,r,s}(x, y; p, q) = (-1)^m J_{m,n,r,s}(x, y; p, q), \quad (27)$$

$$J_{m,-n,r,s}(x, y; p, q) = (-1)^n J_{m,n,r,s}(x, y; p, q), \quad (28)$$

$$J_{m,n,-r,s}(x, y; p, q) = (-1)^r J_{m,n,r,s}(x, y; p, q), \quad (29)$$

$$J_{m,n,r,-s}(x, y; p, q) = (-1)^s J_{m,n,r,s}(x, y; p, q), \quad (30)$$

$$J_{-m,-n,-r,-s}(x, y; p, q) = (-1)^{m+n+r+s} J_{m,n,r,s}(x, y; p, q), \quad (31)$$

Proof. Using (19), we get

$$J_{-m,n,r,s}(x, y; p, q) = \sum_{c=m}^{\infty} \sum_{k=0}^{\infty} (-1)^{c+k} \frac{p^{\binom{-m+c}{2} + \binom{n}{2} + \binom{r}{2} + \binom{s}{2} + \binom{k}{2}}}{[-m+c]_{p,q}! [n+c]_{p,q}! [c]_{p,q}! [r+k]_{p,q}! [s+k]_{p,q}! [k]_{p,q}!} \times \left(\frac{x}{3}\right)^{-m+n+2c} \left(\frac{y}{3}\right)^{r+s+2k}, \quad (32)$$

replace c by $c + m$ in the r.h.s. of (32), we obtain

$$J_{-m,n,r,s}(x, y; p, q) = \sum_{c,k=0}^{\infty} (-1)^{c+m+k} \frac{p^{\binom{c}{2} + \binom{n+c}{2} + \binom{c+m}{2} + \binom{r}{2} + \binom{s}{2} + \binom{k}{2}}}{[c]_{p,q}! [n+c]_{p,q}! [c+m]_{p,q}! [r+k]_{p,q}! [s+k]_{p,q}! [k]_{p,q}!} \times \left(\frac{x}{3}\right)^{m+n+2c} \left(\frac{y}{3}\right)^{r+s+2k}, \quad (33)$$

This generates the necessary relation (27) when definition (19) is applied.

The relations (28), (29), (30), and (31) can also be proven.

Corollary 2.2.

The function $J_{m,n}(x, y; A, B; p, q)$ satisfies the relations:

$$J_{m,n,r,s}(-x, y; p, q) = (-1)^{m+n} J_{m,n,r,s}(x, y; p, q), \quad (34)$$

$$J_{m,n,r,s}(x, -y; p, q) = (-1)^{r+s} J_{m,n,r,s}(x, y; p, q), \quad (35)$$

$$J_{m,n,r,s}(-x, -y; p, q) = (-1)^{m+n+r+s} J_{m,n,r,s}(x, y; p, q). \quad (36)$$

Proof. Since,

$$J_{m,n,r,s}(-x, y; p, q) = \left(-\frac{x}{3}\right)^{m+n} \left(\frac{y}{3}\right)^{r+s} \times \sum_{c,k=0}^{\infty} \frac{(-1)^{c+k} p^{\binom{m+c}{2} + \binom{n+c}{2} + \binom{r}{2} + \binom{s}{2} + \binom{k}{2}}}{[m+c]_{p,q}! [n+c]_{p,q}! [c]_{p,q}! [r+k]_{p,q}! [s+k]_{p,q}! [k]_{p,q}!} \times \left(-\frac{x}{3}\right)^{2c} \left(\frac{y}{3}\right)^{2k} = (-1)^{m+n} \left(\frac{x}{3}\right)^{m+n} \left(\frac{y}{3}\right)^{r+s} \times \sum_{c,k=0}^{\infty} \frac{(-1)^{c+k} p^{\binom{m+c}{2} + \binom{n+c}{2} + \binom{r}{2} + \binom{s}{2} + \binom{k}{2}}}{[m+c]_{p,q}! [n+c]_{p,q}! [c]_{p,q}! [r+k]_{p,q}! [s+k]_{p,q}! [k]_{p,q}!}$$

$$\times \left(\frac{x}{3}\right)^{2c} \left(\frac{y}{3}\right)^{2k}$$

This results in relation (34) in light of (19).

It is also possible to verify the relations (33) and (34).

It is also possible to prove the relations (35) and (36).

Corollary 2.3.

By relation (7), the series expansions of the (p, q) - Humbert function $J_{m,n,r,s}(x, y; p, q)$ is defined by

$$J_{m,n,r,s}(x, y; p, q) = \left(\frac{x}{3}\right)^{m+n} \left(\frac{y}{3}\right)^{r+s} \times \sum_{c,k=0}^{\infty} \left\{ (-1)^{c+k} \frac{q^{\binom{m+c}{2} + \binom{n+c}{2} + \binom{r}{2} + \binom{s}{2} + \binom{k}{2}}}{[m+c]_{p,q}! [n+c]_{p,q}! [c]_{p,q}! [r+k]_{p,q}! [s+k]_{p,q}! [k]_{p,q}!} \times \left(\frac{x}{3}\right)^{2c} \left(\frac{y}{3}\right)^{2k} \right\} \quad (37)$$

Which the generating function of above series is

$$E_{p,q} \left(\frac{x}{3} \left(u + t - \frac{1}{ut} \right) \right) \cdot E_{p,q} \left(\frac{y}{3} \left(v + w - \frac{1}{vw} \right) \right) = \sum_{m,n,r,s=-\infty}^{\infty} J_{m,n,r,s}(x, y; p, q) u^m t^n v^r w^s, \quad (38)$$

Recurrence Relations

Recurrence relations play an important role in the study of special functions and their mathematical properties. They provide useful connections between functions with different parameters and help simplify complex mathematical expressions. In this section, we establish recurrence relations for the (p, q) -Humbert functions $J_{m,n,r,s}(x, y; p, q)$. In particular, Theorem (3.1) presents relations involving the partial derivatives of these functions with respect to (x) and (y) . These relations express the derivatives in terms of shifted arguments and indices of the same family of functions. The results are derived by differentiating the generating function and applying the fundamental properties of (p, q) -exponential functions.

Theorem 3.1. The (p, q) -Humbert functions

$J_{m,n,r,s}(x, y; p, q)$ satisfy the following relation:

$$\frac{\partial}{\partial x} J_{m,n,r,s}(x, y; p, q) = \frac{u}{3p^m} J_{m,n,r,s}(px, y; p, q) + \frac{q^{\frac{m}{2} - \frac{n}{2}} t}{3p^{\frac{m}{2} - \frac{n}{2}}} J_{m,n,r,s} \left(p^{\frac{1}{2}} q^{\frac{1}{2}} x, y; p, q \right) - \frac{1}{3} J_{m+1,n+1,r,s}(qx, y; p, q). \quad (39)$$

And

$$\frac{\partial}{\partial y} J_{m,n,r,s}(x, y; p, q) = \frac{v}{3p^m} J_{m,n,r,s}(x, py; p, q) + \frac{q^{\frac{m}{2} - \frac{n}{2}} t}{3p^{\frac{m}{2} - \frac{n}{2}}} J_{m,n,r,s} \left(x, p^{\frac{1}{2}} q^{\frac{1}{2}} y; p, q \right) - \frac{1}{3} J_{m,n,r+1,s+1}(x, qy; p, q). \quad (40)$$

Proof. Differentiating (18) with respect to x and using (12), yields

$$\sum_{m,n,r,s=-\infty}^{\infty} \frac{\partial}{\partial x} J_{m,n,r,s}(x, y; p, q) u^m t^n v^r w^s = \frac{u}{3} e_{p,q} \left[\frac{pxu}{3} \right] \cdot e_{p,q} \left[\frac{pxt}{3} \right] \cdot e_{p,q} \left[-\frac{px}{3ut} \right] \cdot e_{p,q} \left[\frac{yvw}{3} \right] \cdot e_{p,q} \left[\frac{yxu}{3} \right] \cdot e_{p,q} \left[-\frac{y}{3vw} \right] + \frac{t}{3} e_{p,q} \left[\frac{qxu}{3} \right] \cdot e_{p,q} \left[\frac{pxt}{3} \right] \cdot e_{p,q} \left[-\frac{px}{3ut} \right] \cdot e_{p,q} \left[\frac{yvw}{3} \right] \cdot e_{p,q} \left[\frac{yvw}{3} \right] \cdot e_{p,q} \left[-\frac{y}{3vw} \right] - \frac{1}{3ut} e_{p,q} \left[\frac{qxu}{3} \right] \cdot e_{p,q} \left[\frac{qxt}{3} \right] \cdot e_{p,q} \left[-\frac{px}{3ut} \right] \cdot e_{p,q} \left[\frac{yvw}{3} \right] \cdot e_{p,q} \left[\frac{yvw}{3} \right] \cdot e_{p,q} \left[-\frac{y}{3vw} \right]$$

Using (14) in the l.h.s. of the above equation, we obtain

$$\sum_{m,n,r,s=-\infty}^{\infty} \frac{\partial}{\partial x} J_{m,n,r,s}(x, y; p, q) t^m w^n v^r w^s = \frac{u}{3} \sum_{a=0}^{\infty} \frac{p^{\binom{a}{2}}}{[a]_{p,q}!} \left(\frac{pxu}{3} \right)^a \times \sum_{b=0}^{\infty} \frac{p^{\binom{b}{2}}}{[b]_{p,q}!} \left(\frac{pxt}{3} \right)^b \sum_{c=0}^{\infty} \frac{(-1)^c p^{\binom{c}{2}}}{[c]_{p,q}!} \left(\frac{px}{3ut} \right)^c$$

$$\begin{aligned}
 & \times \sum_{i=0}^{\infty} \frac{p^{(i)}_2}{[i]_{p,q}!} \left(\frac{yw}{3}\right)^i \sum_{j=0}^{\infty} \frac{p^{(j)}_2}{[j]_{p,q}!} \left(\frac{yw}{3}\right)^j \\
 & \times \sum_{k=0}^{\infty} \frac{(-1)^k p^{(k)}_2}{[k]_{p,q}!} \left(\frac{y}{3vw}\right)^k + \frac{t}{3} \sum_{a=0}^{\infty} \frac{p^{(a)}_2}{[a]_{p,q}!} \left(\frac{qxu}{3}\right)^a \\
 & \sum_{b=0}^{\infty} \frac{p^{(b)}_2}{[b]_{p,q}!} \left(\frac{pxt}{3}\right)^b \sum_{c=0}^{\infty} \frac{(-1)^c p^{(c)}_2}{[c]_{p,q}!} \left(\frac{px}{3ut}\right)^c \\
 & \times \sum_{i=0}^{\infty} \frac{p^{(i)}_2}{[i]_{p,q}!} \left(\frac{yw}{3}\right)^i \sum_{j=0}^{\infty} \frac{p^{(j)}_2}{[j]_{p,q}!} \left(\frac{yw}{3}\right)^j \\
 & \times \sum_{k=0}^{\infty} \frac{(-1)^k p^{(k)}_2}{[k]_{p,q}!} \left(\frac{y}{3vw}\right)^k \\
 & - \frac{1}{3ut} \sum_{a=0}^{\infty} \frac{p^{(a)}_2}{[a]_{p,q}!} \left(\frac{qxu}{3}\right)^a \sum_{b=0}^{\infty} \frac{p^{(b)}_2}{[b]_{p,q}!} \left(\frac{qxt}{3}\right)^b \\
 & \times \sum_{c=0}^{\infty} \frac{(-1)^c p^{(c)}_2}{[c]_{p,q}!} \left(\frac{px}{3ut}\right)^c \sum_{i=0}^{\infty} \frac{p^{(i)}_2}{[i]_{p,q}!} \left(\frac{yw}{3}\right)^i \\
 & \times \sum_{j=0}^{\infty} \frac{p^{(j)}_2}{[j]_{p,q}!} \left(\frac{yw}{3}\right)^j \sum_{k=0}^{\infty} \frac{(-1)^k p^{(k)}_2}{[k]_{p,q}!} \left(\frac{y}{3vw}\right)^k \\
 & = \frac{u}{3} \sum_{a,b=0}^{\infty} \sum_{c=0}^{\infty} (-1)^c \frac{p^{(a)}_2 + \binom{b}{2} + \binom{c}{2} + a+b+c}{[a]_{p,q}! [b]_{p,q}! [c]_{p,q}!} \left(\frac{x}{3}\right)^{a+b+c} \\
 & \times u^{a-c} t^{b-c} \\
 & \times \sum_{i,j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{p^{(i)}_2 + \binom{j}{2} + \binom{k}{2}}{[i]_{p,q}! [j]_{p,q}! [k]_{p,q}!} \left(\frac{y}{3}\right)^{i+j+k} v^{i-k} w^{j-k} \\
 & + \frac{t}{3} \sum_{a,b=0}^{\infty} \sum_{c=0}^{\infty} (-1)^c \frac{p^{(a)}_2 + \binom{b}{2} + \binom{c}{2} + b+c}{[a]_{p,q}! [b]_{p,q}! [c]_{p,q}!} \left(\frac{x}{3}\right)^{a+b+c} u^{a-c} t^{b-c} \\
 & \times \sum_{i,j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{p^{(i)}_2 + \binom{j}{2} + \binom{k}{2}}{[i]_{p,q}! [j]_{p,q}! [k]_{p,q}!} \left(\frac{y}{3}\right)^{i+j+k} v^{i-k} w^{j-k} \\
 & - \frac{1}{3ut} \sum_{a,b=0}^{\infty} \sum_{c=0}^{\infty} (-1)^c \frac{p^{(a)}_2 + \binom{b}{2} + \binom{c}{2} + c}{[a]_{p,q}! [b]_{p,q}! [c]_{p,q}!} \left(\frac{x}{3}\right)^{a+b+c} u^{a-c} t^{b-c} \\
 & \times \sum_{i,j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{p^{(i)}_2 + \binom{j}{2} + \binom{k}{2}}{[i]_{p,q}! [j]_{p,q}! [k]_{p,q}!} \left(\frac{y}{3}\right)^{i+j+k} \\
 & \times v^{i-k} w^{j-k}. \quad (41)
 \end{aligned}$$

Replace a, b, i and j by $m+c, n+c, r+k$ and $s+k$ respectively in the right-hand side of equation (41), we get

$$\begin{aligned}
 & \sum_{m,n,r,s=-\infty}^{\infty} \frac{\partial}{\partial x} J_{m,n,r,s}(x, y; p, q) u^m t^n v^r w^s \\
 & = \frac{u}{3} \sum_{m,n=-\infty}^{\infty} \sum_{c=0}^{\infty} (-1)^c \frac{p^{(m+c)}_2 + \binom{n+c}{2} + \binom{c}{2} + m+n+3c}{[m+c]_{p,q}! [n+c]_{p,q}! [c]_{p,q}!} \\
 & \times \left(\frac{x}{3}\right)^{m+n+2c} u^m t^n \\
 & \times \sum_{r,s=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{p^{(r+k)}_2 + \binom{s+k}{2} + \binom{k}{2}}{[r+k]_{p,q}! [s+k]_{p,q}! [k]_{p,q}!} \\
 & \times \left(\frac{y}{3}\right)^{r+s+2k} v^r w^s \\
 & + \frac{t}{3} \sum_{m,n=-\infty}^{\infty} \sum_{c=0}^{\infty} (-1)^c \frac{p^{(m+c)}_2 + \binom{n+c}{2} + \binom{c}{2} + n+2c}{[m+c]_{p,q}! [n+c]_{p,q}! [c]_{p,q}!} q^{m+c} \\
 & \times \left(\frac{x}{3}\right)^{m+n+2c} u^m t^n \\
 & \times \sum_{r,s=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{p^{(r+k)}_2 + \binom{s+k}{2} + \binom{k}{2}}{[r+k]_{p,q}! [s+k]_{p,q}! [k]_{p,q}!} \\
 & \times \left(\frac{y}{3}\right)^{r+s+2k} v^r w^s \\
 & - \frac{1}{3ut} \sum_{m,n=-\infty}^{\infty} \sum_{c=0}^{\infty} (-1)^c \frac{p^{(m+c)}_2 + \binom{n+c}{2} + \binom{c}{2} + c}{[m+c]_{p,q}! [n+c]_{p,q}! [c]_{p,q}!} q^{m+n+2c} \\
 & \times \left(\frac{x}{3}\right)^{m+n+2c} u^m t^n \\
 & \times \sum_{r,s=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{p^{(r+k)}_2 + \binom{s+k}{2} + \binom{k}{2}}{[r+k]_{p,q}! [s+k]_{p,q}! [k]_{p,q}!} \\
 & \times \left(\frac{y}{3}\right)^{r+s+2k} v^r w^s \\
 & \sum_{m,n,r,s=-\infty}^{\infty} \frac{\partial}{\partial x} J_{m,n,r,s}(x, y; p, q) t^m w^n v^r w^s
 \end{aligned}$$

$$\begin{aligned}
 & = \frac{u}{3p^{\frac{m}{2}+\frac{n}{2}}} \sum_{m,n,r,s=-\infty}^{\infty} J_{m,n,r,s} \left(p^{\frac{3}{2}} x, y; p, q\right) t^m w^n v^r w^s \\
 & + \frac{q^{\frac{m}{2}+\frac{n}{2}} t}{3p^m} \sum_{m,n,r,s=-\infty}^{\infty} J_{m,n,r,s} \left(p q^{\frac{1}{2}} x, y; p, q\right) t^m w^n v^r w^s \\
 & - \frac{p^{\frac{m}{2}+\frac{n}{2}}}{3ut} \sum_{m,n,r,s=-\infty}^{\infty} J_{m,n,r,s} \left(p^{\frac{1}{2}} q x, y; p, q\right) t^m w^n v^r w^s. \quad (42)
 \end{aligned}$$

By equating the coefficients of $t^m w^n v^r w^s$ in (42), we get relation (39).

In same way, we get the relation (40).

Theorem 3.2. The polynomials sequence $J_{m,n,r,s}(x, y; p, q)$ satisfies the next recurrence relation

$$\begin{aligned}
 & [m+1]_{p,q} J_{m+1,n,r,s}(x, y; p, q) \\
 & = \frac{x p^{\frac{m}{2}+\frac{n}{2}}}{3 q^{\frac{m}{2}+\frac{n}{2}}} J_{m,n,r,s} \left(p^{\frac{1}{2}} q^{\frac{1}{2}} x, y; p, q\right) \\
 & + \frac{x p^{-m}}{3 u^2 t} J_{m,n,r,s}(p x, y; p, q), \quad (43)
 \end{aligned}$$

$$\begin{aligned}
 & [n+1]_{p,q} J_{m,n+1,r,s}(x, y; p, q) = \\
 & \frac{x p^{\frac{m}{2}+\frac{n}{2}}}{3 q^{\frac{m}{2}+\frac{n}{2}}} J_{m,n,r,s} \left(p^{\frac{1}{2}} q^{\frac{1}{2}} x, y; p, q\right) \\
 & + \frac{x p^{-m}}{3 u t^2} J_{m,n,r,s}(p x, y; p, q), \quad (44)
 \end{aligned}$$

$$\begin{aligned}
 & [r+1]_{p,q} J_{m,n,r+1,s}(x, y; p, q) \\
 & = \frac{y p^{\frac{m}{2}+\frac{n}{2}}}{3 q^{\frac{m}{2}+\frac{n}{2}}} J_{m,n,r,s} \left(x, p^{\frac{1}{2}} q^{\frac{1}{2}} y; p, q\right) \\
 & + \frac{x p^{-m}}{3 v^2 t} J_{m,n,r,s}(x, p y; p, q), \quad (45)
 \end{aligned}$$

And

$$\begin{aligned}
 & [s+1]_{p,q} J_{m,n,r,s+1}(x, y; p, q) \\
 & = \frac{y p^{\frac{m}{2}+\frac{n}{2}}}{3 q^{\frac{m}{2}+\frac{n}{2}}} J_{m,n,r,s} \left(x, p^{\frac{1}{2}} q^{\frac{1}{2}} y; p, q\right) \\
 & + \frac{x p^{-m}}{3 v w^2} J_{m,n,r,s}(x, p y; p, q), \quad (46)
 \end{aligned}$$

Proof. Differentiating (18) with respect to u and using (11), we find

$$\begin{aligned}
 & \sum_{m,n,r,s=-\infty}^{\infty} \frac{\partial}{\partial u} J_{m,n,r,s}(x, y; p, q) u^m t^n v^r w^s \\
 & = \frac{x}{3} e_{p,q} \left[\frac{pxu}{3}\right] \cdot e_{p,q} \left[\frac{xt}{3}\right] \cdot e_{p,q} \left[-\frac{qx}{3ut}\right] \cdot e_{p,q} \left[\frac{yw}{3}\right] \\
 & \cdot e_{p,q} \left[\frac{yw}{3}\right] \cdot e_{p,q} \left[-\frac{y}{3vw}\right] + \frac{x}{3u^2 t} e_{p,q} \left[\frac{pxu}{3}\right] \cdot e_{p,q} \left[\frac{xt}{3}\right] \\
 & \cdot e_{p,q} \left[-\frac{px}{3ut}\right] \cdot e_{p,q} \left[\frac{yw}{3}\right] \cdot e_{p,q} \left[\frac{yw}{3}\right] \cdot e_{p,q} \left[-\frac{y}{3vw}\right]
 \end{aligned}$$

By using relation (6), we obtain

$$\begin{aligned}
 & \sum_{m,n,s,r=-\infty}^{\infty} [m]_{p,q} J_{m,n,r,s}(x, y; p, q) u^{m-1} t^n v^r w^s \\
 & = \frac{x}{3} \sum_{a=0}^{\infty} \frac{p^{(a)}_2}{[a]_{p,q}!} \left(\frac{pxu}{3}\right)^a \sum_{b=0}^{\infty} \frac{p^{(b)}_2}{[b]_{p,q}!} \left(\frac{xt}{3}\right)^b \\
 & \times \sum_{c=0}^{\infty} \frac{(-1)^c p^{(c)}_2}{[c]_{p,q}!} \left(\frac{qx}{3ut}\right)^c \\
 & \times \sum_{i=0}^{\infty} \frac{p^{(i)}_2}{[i]_{p,q}!} \left(\frac{yw}{3}\right)^i \sum_{j=0}^{\infty} \frac{p^{(j)}_2}{[j]_{p,q}!} \left(\frac{yw}{3}\right)^j \\
 & \times \sum_{k=0}^{\infty} \frac{(-1)^k p^{(k)}_2}{[k]_{p,q}!} \left(\frac{y}{3vw}\right)^k \\
 & + \frac{x}{3u^2 t} \sum_{a=0}^{\infty} \frac{p^{(a)}_2}{[a]_{p,q}!} \left(\frac{pxu}{3}\right)^a \sum_{b=0}^{\infty} \frac{p^{(b)}_2}{[b]_{p,q}!} \left(\frac{xt}{3}\right)^b \\
 & \times \sum_{c=0}^{\infty} \frac{(-1)^c p^{(c)}_2}{[c]_{p,q}!} \left(\frac{px}{3ut}\right)^c \times \sum_{i=0}^{\infty} \frac{p^{(i)}_2}{[i]_{p,q}!} \left(\frac{yw}{3}\right)^i \\
 & \sum_{j=0}^{\infty} \frac{p^{(j)}_2}{[j]_{p,q}!} \left(\frac{yw}{3}\right)^j \times \sum_{k=0}^{\infty} \frac{(-1)^k p^{(k)}_2}{[k]_{p,q}!} \left(\frac{y}{3vw}\right)^k \\
 & = \frac{x}{3} \sum_{a,b=0}^{\infty} \sum_{c=0}^{\infty} (-1)^c \frac{p^{(a)}_2 + \binom{b}{2} + \binom{c}{2} + a}{[a]_{p,q}! [b]_{p,q}! [c]_{p,q}!} \left(\frac{x}{3}\right)^{a+b+c} \times \\
 & u^{a-c} t^{b-c} \sum_{i,j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{p^{(i)}_2 + \binom{j}{2} + \binom{k}{2}}{[i]_{p,q}! [j]_{p,q}! [k]_{p,q}!} \left(\frac{y}{3}\right)^{i+j+k}
 \end{aligned}$$

$$\begin{aligned} & \times v^{i-k} w^{j-k} \\ & + \frac{x}{3u^2 t} \sum_{a,b=0}^{\infty} \sum_{c=0}^{\infty} (-1)^c \frac{p^{\binom{a}{2}+\binom{b}{2}+\binom{c}{2}+a+c}}{[a]_{p,q}! [b]_{p,q}! [c]_{p,q}!} \left(\frac{x}{3}\right)^{a+b+c} u^{a-c} t^{b-c} \\ & \times \sum_{i,j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{p^{\binom{i}{2}+\binom{j}{2}+\binom{k}{2}}}{[i]_{p,q}! [j]_{p,q}! [k]_{p,q}!} \left(\frac{y}{3}\right)^{i+j+k} v^{i-k} w^{j-k} \end{aligned}$$

Replace a, b, i and j by $m+c, n+c, r+k$ and $s+k$ respectively in the right-hand side of equation (41), we get

$$\begin{aligned} & \sum_{m,n,s,r=-\infty}^{\infty} [m+1]_{p,q} J_{m+1,n,r,s}(x,y;p,q) u^m t^n v^r w^s \\ & = \frac{x}{3} \sum_{m,n=-\infty}^{\infty} \sum_{c=0}^{\infty} (-1)^c \frac{p^{\binom{m+c}{2}+\binom{n+c}{2}+\binom{c}{2}+m+c}}{[m+c]_{p,q}! [n+c]_{p,q}! [c]_{p,q}!} \left(\frac{x}{3}\right)^{m+n+2c} \\ & \times u^m t^n \sum_{r,s=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{p^{\binom{r+k}{2}+\binom{s+k}{2}+\binom{k}{2}}}{[r+k]_{p,q}! [s+k]_{p,q}! [k]_{p,q}!} \\ & \times \left(\frac{y}{3}\right)^{r+s+2k} v^r w^s \\ & + \frac{x}{3u^2 t} \sum_{m,n=-\infty}^{\infty} \sum_{c=0}^{\infty} (-1)^c \frac{p^{\binom{m+c}{2}+\binom{n+c}{2}+\binom{c}{2}+n+2c}}{[m+c]_{p,q}! [n+c]_{p,q}! [c]_{p,q}!} \\ & \times \left(\frac{x}{3}\right)^{m+n+2c} u^m t^n \\ & \sum_{r,s=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{p^{\binom{r+k}{2}+\binom{s+k}{2}+\binom{k}{2}}}{[r+k]_{p,q}! [s+k]_{p,q}! [k]_{p,q}!} \left(\frac{y}{3}\right)^{r+s+2k} v^r w^s \end{aligned}$$

Comparing of both sides, we get the relation (43).

Similarly, way differentiating (18) with respect to t, v and w , we find relations (44), (45) and (46) respectively.

Theorem 3.3. For polynomial $J_{m,n}(x, y; A, B; p, q)$ holds

$$\begin{aligned} & \frac{p^{\binom{m}{2}+\binom{n}{2}+\binom{r}{2}+\binom{s}{2}}}{[m]_{p,q}! [n]_{p,q}! [r]_{p,q}! [s]_{p,q}!} \left(\frac{x}{3}\right)^{m+n} \left(\frac{y}{3}\right)^{r+s} = \\ & \sum_{c,k=0}^{\infty} \frac{p^{\binom{c}{2}+\binom{k}{2}}}{[c]_{p,q}! [k]_{p,q}!} \left(\frac{x}{3}\right)^c \left(\frac{y}{3}\right)^k J_{m+c,n+c,r+k,s+k}(x, y; p, q) \quad (47) \end{aligned}$$

Proof. Using generating function of polynomials $J_{m,n,r,s}(x, y; p, q)$ and definition expression for functions

$$\begin{aligned} & e_{p,q} \left[\frac{xu}{3} \right], e_{p,q} \left[\frac{xt}{3} \right], e_{p,q} \left[\frac{yv}{3} \right], e_{p,q} \left[\frac{yw}{3} \right], e_{p,q} \left[\frac{x}{3ut} \right], \\ & \text{and } e_{p,q} \left[\frac{y}{3vw} \right] \text{ we have} \\ & e_{p,q} \left[\frac{xu}{3} \right] \cdot e_{p,q} \left[\frac{xt}{3} \right] \cdot e_{p,q} \left[\frac{yv}{3} \right] \cdot e_{p,q} \left[\frac{yw}{3} \right] = e_{p,q} \left[\frac{x}{3ut} \right] \\ & \cdot e_{p,q} \left[\frac{y}{3vw} \right] \times \sum_{m,n,s,r=-\infty}^{\infty} J_{m,n,r,s}(x, y; p, q) u^m t^n v^r w^s \end{aligned}$$

Using the relation (1.6)

$$\begin{aligned} & \sum_{m=0}^{\infty} \frac{p^{\binom{m}{2}}}{[m]_{p,q}!} \left(\frac{xu}{3}\right)^m \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}}{[n]_{p,q}!} \left(\frac{xt}{3}\right)^n \sum_{r=0}^{\infty} \frac{p^{\binom{r}{2}}}{[r]_{p,q}!} \left(\frac{yv}{3}\right)^r \\ & \times \sum_{s=0}^{\infty} \frac{p^{\binom{s}{2}}}{[s]_{p,q}!} \left(\frac{yw}{3}\right)^s \\ & = \sum_{c=0}^{\infty} \frac{p^{\binom{c}{2}}}{[c]_{p,q}!} \left(\frac{x}{3ut}\right)^c \sum_{k=0}^{\infty} \frac{p^{\binom{k}{2}}}{[k]_{p,q}!} \left(\frac{y}{3vw}\right)^k \\ & \times \sum_{m,n,s,r=-\infty}^{\infty} J_{m,n,r,s}(x, y; p, q) u^m t^n v^r w^s \\ & \sum_{m,n,r,s=0}^{\infty} \frac{p^{\binom{m}{2}+\binom{n}{2}+\binom{r}{2}+\binom{s}{2}}}{[m]_{p,q}! [n]_{p,q}! [r]_{p,q}! [s]_{p,q}!} \left(\frac{x}{3}\right)^{m+n} \\ & \times \left(\frac{y}{3}\right)^{r+s} u^m t^n v^r w^s = \sum_{c,k=0}^{\infty} \frac{p^{\binom{c}{2}+\binom{k}{2}}}{[c]_{p,q}! [k]_{p,q}!} \left(\frac{x}{3}\right)^c \left(\frac{y}{3}\right)^k \end{aligned}$$

$$\begin{aligned} & \times \sum_{m,n,s,r=-\infty}^{\infty} J_{m,n,r,s}(x, y; p, q) u^{m-c} t^{n-c} v^{r-k} w^{s-k} \\ & = \sum_{c,k=0}^{\infty} \frac{p^{\binom{c}{2}+\binom{k}{2}}}{[c]_{p,q}! [k]_{p,q}!} \left(\frac{x}{3}\right)^c \left(\frac{y}{3}\right)^k \\ & \times \sum_{m,n,s,r=0}^{\infty} J_{m+c,n+c,r+k,s+k}(x, y; p, q) u^m t^n v^r w^s \end{aligned}$$

Comparing of the coefficients of $u^m t^n v^r w^s$ of the above equation, we obtain the required relation (47).

Conclusion

In this paper, we introduced a (p, q) -analogue of the two-variable Humbert function within the framework of post-quantum calculus. By employing generating-function techniques and the properties of (p, q) -exponential functions,

we derived an explicit series representation of the proposed function and established its associated generating functions. Furthermore, several fundamental properties were investigated, including symmetry relations, parity identities, and limiting relations connecting the proposed function to its classical and (q) -analogues.

We also derived differential and recurrence relations involving the parameters and arguments of the newly defined function. In addition, a polynomial identity was established, providing further insight into the algebraic structure of the proposed family of functions. These results extend selected properties of classical Humbert functions to the (p, q) -calculus setting and enrich the existing theory of multivariable special functions.

The findings of this study provide a theoretical foundation for further investigations into the analytical properties, computational representations, and potential applications of (p, q) -Humbert functions. Future research may focus on developing symbolic algorithms, numerical approximation methods, and application-oriented models in mathematical physics and engineering computation, as well as exploring further connections with other families of generalized special functions.

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